Homework 2 solution
Math 311, Spring 2009

(A-7) Prove that if \( b > 0 \), then there exist only finitely many positive integers \( n \) such that \( 0 < n \leq b \). Hence the set \( \{ n : n \in \mathbb{N}, n \leq b \} \) is a finite subset of \( \mathbb{N} \), and it has a maximum element if it is nonempty.

For \( b > 0 \), there exists \( k \in \mathbb{N} \) such that \( b < k \). Then for any \( n > k \) and \( n \in \mathbb{N} \), \( n > k > b \). So if \( 0 < n \leq b \) then \( n \in \{ 1, 2, 3, \ldots, k-1 \} \), which is a finite set. So there exist only finitely many positive integers \( n \) such that \( 0 < n \leq b \).

(A-8) It is known that \( \mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \} \) (where \( i^2 = -1 \)) is a field. Prove that \( \mathbb{C} \) is not an ordered field, that is, one cannot define an order relation on \( \mathbb{C} \) which satisfies (O1)-(O5).

If there is such an order, then either \( i < 0 \) or \( i > 0 \) since \( i \neq 0 \). If \( i > 0 \), then \( -1 = i \cdot i > 0 \) from Theorem 3.2. But from \( -1 > 0 \) we can get \(-1 + 1 > 0 + 1 \) that is \( 0 > 1 \), and from \(-1 > 0 \) we also get \( 1 = (-1)^2 > 0 \) from Theorem 3.2. So \( 0 > 1 > 0 \) which is impossible. If \( i < 0 \), then \( i + (-i) < 0 + (-i) \) so \( 0 < -i \), again \(-1 = (-i) \cdot (-i) > 0 \) from Theorem 3.2. Again we obtain \( 0 > 1 > 0 \) which is impossible. Therefore there exists no order for complex field.

(A-9) The floor function is defined by \( \lfloor x \rfloor = \max \{ n \in \mathbb{Z} \mid n \leq x \} \). And the fractional part function is \( \{ x \} = x - \lfloor x \rfloor \). For all \( x \), \( 0 \leq \{ x \} < 1 \). For example, \( \{ 2.3 \} = 2 \), \( \{ 2.3 \} = 0.3 \); and \( \{ -2.3 \} = -0.3 \). Prove that if \( a \) is an irrational number, then

(a) For any \( \varepsilon > 0 \), there exist \( m, n \in \mathbb{N} \) such that \( |na - m| < \varepsilon \);

(b) For any \( a, b \) satisfying \( 0 < a < b < 1 \), there exists \( n \in \mathbb{N} \) such that \( \{ na \} \in (a, b) \).

See http://en.wikipedia.org/wiki/Pigeonhole_principle,
http://www.cms.math.ca/Students/Problems/PigSol.pdf

(4.1-4.4) (b) \((0,1)\): upper bound: 1, 2, 3, lower bound: \(-2, -1, 0\); sup = 1, inf = 0.

(i): \( \bigcap_{n=1}^{\infty} [-1/n, 1 + 1/n] \): upper bound: 2, 3, lower bound: \(-2, -3, -1\). sup = 1, inf = 0.

(n): \( \{ r \in \mathbb{Q} : r^2 < 2 \} \): upper bound: 2, 3, lower bound: \(-2, -3, -4\). sup = \( \sqrt{2} \), inf = \(-\sqrt{2} \).

(v): \( \{ n(\pi/3) \} : n \in \mathbb{N} \} \): upper bound: 1, 2, 3, lower bound: \(-2, -3, -1\). sup = 1, inf = -1.

(4.6) proof: Since \( S \) is not empty, then there exists \( s \in S \). From definition of infimum and supremum, inf \( S \leq s \leq \sup S \). Then inf \( S \leq \sup S \). If inf \( S = \sup S \), then \( S \) has only one element.

(4.8) (a) Given \( a \in T \), then for any \( s \in T \), \( s \leq t \). So \( t \) is an upper bound of \( S \). Similarly any \( s \in S \) is a lower bound of \( T \).

(b) Let \( s_0 = \sup S \) and \( t_0 = \inf T \). Suppose that \( s_0 > t_0 \). Then from Archimedean property, there exists \( \varepsilon > 0 \) such that \( s_0 - t_0 > 4\varepsilon \). Since \( s_0 = \sup S \), then there exists \( s_1 \in S \) such that \( s_1 > s_0 - \varepsilon \), and since \( t_0 = \inf T \), there exists \( t_1 \in T \) such that \( t_1 < t_0 + \varepsilon \). Then \( t_1 < t_0 + \varepsilon < s_0 - \varepsilon < s_1 \) (here \( t_0 + \varepsilon < s_0 - \varepsilon \) is because \( s_0 - t_0 > 4\varepsilon \)). But \( t_1 < s_1 \) contradicts with the assumption that \( s \leq t \) for all \( s \in S \) and \( t \in T \). Hence \( s_0 < t_0 \) holds.

(c) \( S = (0,1], T = [1,2) \); (d) \( S = (0,1), T = (1,2) \).

Alternate proof of (b): Let \( s_0 = \sup S \) and \( t_0 = \inf T \). Suppose that \( s_0 > t_0 \). Then from Archimedean property, there exists \( x \) satisfying \( s_0 > x > t_0 \). Since \( s_0 \) is the smallest upper bound of \( S \) and \( x < s_0 \), then \( x \) is not an upper bound of \( S \) and there exists \( s_1 \in S \) such that \( s_1 > x \). Similarly \( x \) is not a lower bound of \( T \), then there exists \( t_1 \in T \) such that \( t_1 < x \). Hence \( s_1 > t_1 \) but that contradicts with \( s \leq t \) for any \( s \in S \) and \( t \in T \).

(4.14a) Let \( s = \sup S \), \( a = \sup A \) and \( b = \sup B \). First we prove \( s \geq a + b \). If not then we have \( s < a + b \). From Archimedean property, there exists \( \varepsilon > 0 \) such that \( (a + b) - s > 4\varepsilon > 0 \). Since \( a = \sup A \), then there exists \( a_1 \in A \) such that \( a_1 > a - \varepsilon \), and since \( b = \sup B \), there exists \( b_1 \in B \) such that \( b_1 > b - \varepsilon \). Then \( a_1 + b_1 > (a + b) - 2\varepsilon > s + 2\varepsilon > s \). But \( s = \sup S \) then \( s \geq a_2 + b_2 \) for any \( a_2 \in A \) and \( b_2 \in B \). That is a contradiction. Hence \( s \geq a + b \). Second we prove \( s \leq a + b \). If not then we have \( s > a + b \). From Archimedean property, there exists \( \varepsilon > 0 \) such that \( s - (a + b) > 4\varepsilon > 0 \). Since \( s = \sup S \), then there exists \( a_3 \in A \) and \( b_3 \in B \) such that \( (a_3 + b_3) > s - 2\varepsilon \) or \( a_3 + b_3 > s - 2\varepsilon > a + b + 2\varepsilon \). Then either \( a_3 > a + \varepsilon \) or \( b_3 > b + \varepsilon \) holds (otherwise \( a_3 \leq a + \varepsilon \) and \( b_3 \leq b + \varepsilon \) will imply \( a_3 + b_3 \leq a + b + 2\varepsilon \)). But \( a_3 > a + \varepsilon \) contradicts with \( a = \sup A \), and \( b_3 > b + \varepsilon \) contradicts with \( b = \sup B \). Therefore \( s \leq a + b \). Now we must have \( s = a + b \) since \( s \geq a + b \) and \( s \leq a + b \).

(4.15) see solution on page 314

(5.1-5.2) (c) \( [0,\infty), \inf = 0 \) and \( \sup = \infty \); (d) \( (-\infty, \sqrt{8}), \inf = -\infty \) and \( \sup = \sqrt{8} \).

(6.4) From that definition, \( 0^* \cdot 1^* = \mathbb{Q} \), but that is not reasonable since we expect to get \( 0^* \cdot 1^* = \{ r \in \mathbb{Q} : r < 0 \} \).