(26.2) (a) From example 1: \( \sum nx^{n-1} = \frac{1}{(1-x)^2} \), and multiplying another \( x \), we get \( \sum nx^n = \frac{x}{(1-x)^2} \). (b) In \( \sum nx^n = \frac{x}{(1-x)^2} \), let \( x = 1/2 \), then \( \sum n2^{-n} = 2 \). (c) In \( \sum nx^n = \frac{x}{(1-x)^2} \), let \( x = 1/3 \) and \( x = -1/3 \), then \( \sum n3^{-n} = 3/4 \) and \( \sum (-1)^n3^{-n} = -3/16 \).

(26.3) (a) Differentiating \( \sum nx^n = \frac{x}{(1-x)^2} \), we get \( \sum n^2x^{n-1} = \frac{x+1}{(1-x)^3} \). Multiplying another \( x \), we get \( \sum n^2x^n = \frac{x^2+x}{(1-x)^3} \), let \( x = 1/2 \) and \( x = 1/3 \), then \( \sum n^22^{-n} = 6 \) and \( \sum n^23^{-n} = 3/2 \).

(26.4) (a) In \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \), substituting \( x \) by \(-x^2\), we get \( e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} \). (b) Integrating \( e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!} \), and Theorem 26.4, we get \( \int_0^x e^{-t^2} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \int_0^x \frac{2k}{k!} dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \).

(28.1e) \( x = \pm 1 \)

(28.2d) \( r(1) = 7 \), so by definition \( r'(x) = \lim_{x \to 1} \frac{r(x) - 7}{x - 1} = \lim_{x \to 1} \frac{-11(x - 1)}{(2x - 1)(x - 1)} = \lim_{x \to 1} \frac{-11}{2x - 1} = -11 \).

(28.4) (a) \( f_1(x) = x \) is differentiable from definition; \( f_2(x) = 1/x \) is differentiable from quotient rule and \( f_1 \) is differentiable as long as \( x \neq 0 \); \( f_3(x) = \sin(1/x) \) is differentiable from chain rule and \( f_2 \) is differentiable when \( x \neq 0 \); \( f_4(x) = x^2 \) from Example 3; and finally \( f_5(x) = x^2 \sin(1/x) \) is differentiable from product rule and \( f_3, f_4 \) are differentiable.

(28.7) (b) \( f'(0) = \lim_{x \to 0} \frac{f(x)}{x} \). Then \( \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0 \), and \( \lim_{x \to 0^-} \frac{x}{x} = 0 \). So \( f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = 0 \) since the limits from left and right are both 0. (c) \( f'(x) = 2x \) when \( x \geq 0 \) and \( f'(x) = 0 \) when \( x < 0 \). (d) \( f'(x) \) is continuous at \( x = 0 \) since \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0 \), but \( f'(x) \) is not differentiable since \( \lim_{x \to 0^-} f'(x) = 1 \) but \( \lim_{x \to 0^+} f'(x) = 0 \).

(28.8) (a) \( f(0) = 0 \), for \( \varepsilon > 0 \), let \( \delta = \sqrt{\varepsilon} > 0 \). Then when \( |x| < \delta \), \( |f(x) - 0| < \varepsilon \) no matter \( x \) is rational or irrational. (b) For \( x \neq 0 \), if \( x \) is rational, then there exists an irrational sequence \( x_n \to x \), then \( \lim f(x_n) = 0 \neq f(x) = x^2 \); if \( x \) is irrational, then there exists a rational sequence \( x_n \to x \), then \( \lim f(x_n) = x^2 \neq f(x) = 0 \). (c) \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} \). Let \( g(x) = f(x)/x \) for \( x \neq 0 \), then \( g(x) = x \) if \( x \) is rational, and \( g(x) = 0 \) if \( x \) is irrational. Then \( f'(0) = \lim_{x \to 0^-} g(x) = 0 \) since limits from the left and right are all 0.

(A-25) Suppose that \( f_0(x) \) is continuous on \( [a, b] \). Define

\[
f_k(x) = \int_a^x f_{k-1}(t)dt, \quad k = 1, 2, \cdots.
\]

Prove the sequence of the functions \( f_k(x) \) converges uniformly to \( f(x) = 0 \).

**The original problem has a typo. This is correct form.**

Since \( f_0(x) \) is continuous on \( [a, b] \), then \( |f_0(x)| \leq M \) for \( x \in [a, b] \) from Theorem 18.1. Then

\[
|f_1(x)| \leq \int_a^x |f_0(t)|dt \leq M(x-a), \quad |f_2(x)| \leq \int_a^x |f_1(t)|dt \leq \frac{1}{2} M(x-a)^2.
\]

Following this pattern, one can prove that for each \( n \in \mathbb{N} \) and \( x \in [a, b] \),

\[
|f_n(x)| \leq \frac{M}{n!} (x-a)^n \leq \frac{M}{n!} (b-a)^n \text{(defined to be } M_n)\
\]

Then \( 0 \leq |f_n(x)| \leq M_n = \frac{M}{n!} (b-a)^n \), and \( \lim_{n \to \infty} \frac{M(b-a)^n}{n!} = 0 \). So \( f_k(x) \) converges uniformly to \( f(x) = 0 \) from the definition.
(A-27) Prove that there is no differentiable function \( f: \mathbb{R} \to \mathbb{R} \) such that \( f(f(x)) = x^2 - 3x + 3 \) for all \( x \in \mathbb{R} \).

First, \( f(x) \) cannot be monotone. If \( f' > 0 \) or \( f' < 0 \) for all \( x \in \mathbb{R} \), then by differentiating \( f(f(x)) = x^2 - 3x + 3 \), we find \( f'(f(x)) \cdot f'(x) = 2x - 3 \). The function \( 2x - 3 \) is not always positive, but the left hand side will be if \( f \) is monotone. So there exists at least one point \( x_0 \in \mathbb{R} \) such that \( f'(x_0) = 0 \). Substituting \( x = x_0 \) into \( f'(f(x)) \cdot f'(x) = 2x - 3 \), we find \( 2x_0 - 3 = 0 \), so \( x_0 = 3/2 \) is the only critical point of \( f(x) \).

We look for fixed points for \( x = f(x) \) and \( x = f(f(x)) \). Set \( x = x^2 - 3x + 3 \), we can solve \( x = 1 \) and \( x = 3 \). So \( f(1) = 1 \) and \( f(3) = 3 \). It is possible that \( f(1) = 3 \) and \( f(3) = 1 \). In either case, \( f(1) \) and \( f(3) \) are 1 and 3, then by the Mean Value Theorem, there exists \( x_1 \in (1, 3) \) such that \( f(x_1) = 3/2 \) since \( 3/2 \in (1, 3) \). Now substituting \( x = x_1 \) into \( f'(f(x)) \cdot f'(x) = 2x - 3 \), we find \( 2x_1 - 3 = 0 \). So \( x_1 = 3/2 \) and \( f(3/2) = 3/2 \). But we have proved the only possible fixed points are 1 and 3. That is a contradiction. So there exists no differentiable function \( f: \mathbb{R} \to \mathbb{R} \) such that \( f(f(x)) = x^2 - 3x + 3 \).

(A-28) Find the sum of the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \) by using differentiation and/or integration of power series.

Starting from \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \), substituting \( x = -x^2 \), we obtain \( \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \). Subtract 1 from both sides, and multiplying by \(-1\), we get \( \sum_{n=1}^{\infty} (-1)^{n-1} x^{2n} = \frac{x^2}{1+x^2} \). Dividing both sides by \( x^2 \), now we have \( \sum_{n=1}^{\infty} (-1)^{n-1} x^{2n-2} = \frac{1}{1+x^2} \). Finally we integrate it, then \( \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^x t^{2n-2} dt = \int_0^x \frac{1}{1+t^2} dt \), hence

\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1} x^{2n-1} = \tan^{-1}(x). \]

Now let \( x = 1 \) (the series is convergent at \( x = 1 \)), then we obtain

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}.
\]