Final Exam
Math 311, Spring 2009

Take-home part: you can use textbook and notebooks, but not other reference books or electronic resource; and you have to complete it by yourself. Choose 7 problems from the following 10 problems to solve, and each will be counted as extra 2%. (Total 56%+extra 6%, time limit: now — May 4th, 11:30am)

For all the problems, clearly state all assumptions, theorems or known results, and write your proof in complete sentences. When you use a theorem or exercise in textbook, use the notation like “Theorem 3.10(ii)”, “Exercise 10.11(c)”.

1. Define a function $f(x)$ by
   
   $$f(x) = \begin{cases} 
   x, & \text{if } x \in \mathbb{Q}; \\
   -x, & \text{if } x \not\in \mathbb{Q}.
   \end{cases}$$

   (a) Determine at which $x \in \mathbb{R}$, $f(x)$ is continuous. Prove your claim.
   (b) Determine at which $x \in \mathbb{R}$, $f(x)$ is differentiable. Prove your claim.

2. (a) Suppose $f$ is a real-valued function on $\mathbb{R}$ which satisfies
   
   $$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = 0$$

   for all $x \in \mathbb{R}$. Does this imply that $f$ is differentiable at each $x \in \mathbb{R}$? If yes, provide a proof; if no, provide a counter-example.
   (b) Suppose $f$ is a real-valued function on $\mathbb{R}$ which satisfies
   
   $$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

   for all $x \in \mathbb{R}$. Does this imply that $f$ is continuous at each $x \in \mathbb{R}$? If yes, provide a proof; if no, provide a counter-example.

3. Suppose $f : I \to \mathbb{R}$ is continuous on $I$, where $I = [a, b]$ is closed and bounded. Suppose that for each $n \in \mathbb{N}$, there exists a point $x_n \in I$ such that $M - \frac{1}{n} < f(x_n) < M$. Prove that there exists $c \in I$ such that $f(c) = M$.

4. Let $f$ be continuous on $[0, \infty)$. Suppose $\lim_{x \to \infty} f(x) = L$ and $L$ is finite.
   (a) Prove that $f(x)$ is bounded on $[0, \infty)$.
   (b) Prove that $f$ is uniformly continuous on $[0, \infty)$.

5. Suppose that $p(x)$ is a polynomial, i.e. $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for $a_i \in \mathbb{R}$ ($0 \leq i \leq n$), $a_n \neq 0$. Prove that there exists $x_0 > 0$ such that $p(x)$ is monotone in $[x_0, \infty)$.

6. Let $f(x)$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Suppose that $f(0) = f(1) = 0$ and that there is an $x_0 \in (0, 1)$ such that $f(x_0) = 1$. Prove that $|f'(y)| \geq 2$ for some $y \in (0, 1)$.
7. Consider the power series \( f(x) = \sum_{n=2}^{\infty} \frac{x^n}{n(n-1)} \).

(a) Determine the convergence interval of the power series.

(b) Derive an explicit formula for the power series, and use it to evaluate \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} \).

8. Let \( f_n(x) \) be a sequence of functions given by
\[
f_n(x) = \frac{2x^3}{1 + nx^4}, \quad x \in \mathbb{R}.
\]

(a) Find \( f(x) = \lim_{n \to \infty} f_n(x) \).

(b) Does \( f_n(x) \) converge uniformly to \( f(x) \) on \( (-\infty, \infty) \)?

(c) Does \( f_n(x) \) converge uniformly to \( f(x) \) on \([0, 1]\)?

9. Suppose that \( f(x) \) is a continuous function on \([a, b]\), and for any other continuous \( g(x) \) on \([a, b]\), we have \( \int_{a}^{b} f(x)g(x)\,dx = 0 \). Prove that \( f(x) = 0 \) for any \( x \in [a, b] \).

10. (a) If \( |f(x)| \) is integrable on \([a, b]\), is \( f(x) \) also integrable on \([a, b]\)? Prove or give a counterexample.

(b) If \( [f(x)]^2 \) is integrable, is \( f(x) \) also integrable on \([a, b]\)? Prove or give a counterexample.