Midterm Exam
Math 311, Spring 2009

Take-home part: you can use textbook and notebooks, but not other reference books or electronic resource; and you have to complete it by yourself. Choose 7 problems from the following 10 problems to solve, and each is 8% (total 56% of the midterm, the in-class part is 44%). You can solve the other 3 problems, and each will be counted as extra 2%. (Total 56%+extra 6%, time limit: now — March 5th, 11am)

For all the problems, clearly state all assumptions, theorems or known results, and write your proof in complete sentences. When you use a theorem or exercise in textbook, use the notation like “Theorem 3.10(ii)”, “Exercise 10.11(c)”.

1. Prove \( \sqrt{n-1} + \sqrt{n+1} \) is irrational for every \( n \in \mathbb{N} \).

2. Using only the axioms of ordered field on page 13 to prove: if \( a \leq b \) and \( c \leq d \), then \( ad + bc \leq ac + bd \). Please state which axiom(s) you use in each step.

3. Let \( A \) be a non-empty subset of the real numbers, which \( \sup A = 3 \) and \( \inf A = -2 \). Find the \( \sup B \) and \( \inf B \) where \( B = \{ \frac{1}{x+5} - y^3 : x, y \in A \} \). You do not need to prove your answers by definition.

4. Prove that \( \lim_{n \to \infty} \frac{n^2 - n + 1}{2n^2 + n - 10} = \frac{1}{2} \) by using the definition of the limit of sequence. Do not use limit theorems, and explicitly show the choice of \( N \).

5. Assume that \( (a_n) \) and \( (b_n) \) are Cauchy sequences. Use triangle inequality and the definition of Cauchy sequence to show that the sequence \( (c_n) \) defined by \( c_n = |a_n - b_n| \) is also Cauchy.

6. Prove that for any real number \( x \), there exists a sequence \( (x_n) \) of rational numbers, which is monotone increasing, and converges to \( x \). (Hint: compare with Exercise 10.7, and notice here we need it to be increasing not nondecreasing)

7. Let \( (s_n) \) be a bounded sequence of real numbers, and let \( k \) be a positive real number. Prove that \( \lim \sup (ks_n) = k \lim \sup s_n \). (Hint: to prove \( a = b \), you need to prove \( a \leq b \) and \( a \geq b \).)

8. (a) Give an example of a positive sequence \( (a_n) \) satisfying \( \lim \inf \frac{a_{n+1}}{a_n} < \lim \sup \frac{a_{n+1}}{a_n} \).

(b) Give an example of a positive sequence \( (a_n) \) satisfying

\[
\lim \inf \frac{a_{n+1}}{a_n} < \lim \inf (a_n)^{1/n} < \lim \sup (a_n)^{1/n} < \lim \sup \frac{a_{n+1}}{a_n}.
\]

9. Determine whether the following series converge. Clearly state which criterion(s) you use.

(i) \( \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2 + 2n} \); (ii) \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \); (iii) \( \sum_{n=1}^{\infty} (\sqrt{3n+1} - \sqrt{3n}) \).

10. Use the definition of convergent series to prove that if (i) \( \sum a_n \) and \( \sum b_n \) are both series of positive terms, (ii) \( \sum b_n \) is convergent, and (iii) \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \), then \( \sum a_n \) is also convergent.
1. (16 points) Choose TRUE or FALSE for the following statements. If you choose FALSE, please give an example for which the statement does not hold.

(a) T  F  If $A$ and $B$ are two bounded sets of real numbers, then $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

(b) T  F  Let $(s_n)$ be a bounded sequence of real numbers and $\lim \sup s_n = b < 0$. Then there exists $N$ such that when $n > N$, then $s_n$ is negative.

(c) T  F  If $\lim s_n$ exists, then $\lim s_n^2$ also exists.

(d) T  F  If $\lim s_n^2$ exists, then $\lim s_n$ also exists.

(e) T  F  If a series $\sum a_n^2$ is convergent, then $\sum a_n$ is also convergent.

(f) T  F  If a series $\sum a_n$ is convergent, then $\sum a_n^2$ is also convergent.

(g) T  F  If $\sum a_n$ is convergent, then $\lim \sup |a_{n+1}/a_n| < 1$.

(h) T  F  If the sequence $(s_n)$ satisfies $s_n > 0$, $\lim s_n = 0$ and $(s_n)$ is decreasing, then the infinite series $\sum s_n$ is convergent.

2. (6 points) Let $s_n = \cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right) + (-1)^n$.

(a) Find the set of subsequential limits of $(s_n)$.

(b) Find $\sup\{s_n\}$ and $\inf\{s_n\}$.

(c) Find a subsequence of $(s_n)$ which converges to $\sup\{s_n\}$. You need to give the selection function $\sigma(k)$ for the subsequence.
3. (10 points) Let \((s_n)\) be a sequence of real numbers.
   
   (a) Define what it means for \((s_n)\) to be a Cauchy sequence.
   
   (b) Give the negation of the statement “\((s_n)\) is Cauchy” according to your definition in part (a).
   
   (c) Prove if \(\lim s_n = s\), then \((s_n)\) is a Cauchy sequence.

4. (6 points) Suppose that \((a_n)\) and \((b_n)\) are two sequences satisfying \(a_n \leq b_n\) for any \(n \in \mathbb{N}\), and \((a_n)\), \((b_n)\) are both convergent. Prove that \(\lim a_n \leq \lim b_n\) by using the definition of the limit. (You cannot apply any theorems or exercises in textbook which directly imply this statement.)

5. (6 points) Define a sequence \((x_n)\) by \(x_1 = 2\), and \(x_{n+1} = 2 - \frac{1}{x_n}\).
   
   (a) Prove \(x_n > 1\) for all \(n \in \mathbb{N}\).
   
   (b) Prove \((x_n)\) is decreasing.
   
   (c) From monotone convergent theorem, \((x_n)\) is convergent. What is the limit of \((x_n)\)?