Midterm Exam
Math 311, Spring 2009

Take-home part: you can use textbook and notebooks, but not other reference books or electronic resource; and you have to complete it by yourself. Choose 7 problems from the following 10 problems to solve, and each is 8% (total 56% of the midterm, the in-class part is 44%). You can solve the other 3 problems, and each will be counted as extra 2%. (Total 56%+extra 6%, time limit: now — March 5th, 11am)

For all the problems, clearly state all assumptions, theorems or known results, and write your proof in complete sentences. When you use a theorem or exercise in textbook, use the notation like “Theorem 3.10(ii)”, “Exercise 10.11(c)”.

1. Prove \( \sqrt{n-1} + \sqrt{n+1} \) is irrational for every \( n \in \mathbb{N} \).

2. Using only the axioms of ordered field on page 13 to prove: if \( a \leq b \) and \( c \leq d \), then \( ad + bc \leq ac + bd \). Please state which axiom(s) you use in each step.

3. Let \( A \) be a non-empty subset of the real numbers, which \( \sup A = 3 \) and \( \inf A = -2 \). Find the \( \sup B \) and \( \inf B \) where \( B = \{ \frac{1}{x + 5} - y^3 : x, y \in A \} \). You do not need to prove your answers by definition.

4. Prove that \( \lim_{n \to \infty} \frac{n^2 - n + 1}{2n^2 + n - 10} = \frac{1}{2} \) by using the definition of the limit of sequence. Do not use limit theorems, and explicitly show the choice of \( N \).

5. Assume that \( (a_n) \) and \( (b_n) \) are Cauchy sequences. Use triangle inequality and the definition of Cauchy sequence to show that the sequence \( (c_n) \) defined by \( c_n = |a_n - b_n| \) is also Cauchy.

6. Prove that for any real number \( x \), there exists a sequence \( (x_n) \) of rational numbers, which is monotone increasing, and converges to \( x \). (Hint: compare with Exercise 10.7, and notice here we need it to be increasing not nondecreasing)

7. Let \( (s_n) \) be a bounded sequence of real numbers, and let \( k \) be a positive real number. Prove that \( \lim \sup (ks_n) = k \lim \sup s_n \). (Hint: to prove \( a = b \), you need to prove \( a \leq b \) and \( a \geq b \).)

8. (a) Give an example of a positive sequence \( (a_n) \) satisfying \( \lim \inf \frac{a_{n+1}}{a_n} < \lim \sup \frac{a_{n+1}}{a_n} \).

(b) Give an example of a positive sequence \( (a_n) \) satisfying
\[
\lim \inf \frac{a_{n+1}}{a_n} < \lim \inf (a_n)^{1/n} < \lim \sup (a_n)^{1/n} < \lim \sup \frac{a_{n+1}}{a_n}.
\]

9. Determine whether the following series converge. Clearly state which criterion(s) you use.

(i) \( \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2 + 2n} \); (ii) \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \); (iii) \( \sum_{n=1}^{\infty} (\sqrt{3n + 1} - \sqrt{3n}) \).

10. Use the definition of convergent series to prove that if (i) \( \sum a_n \) and \( \sum b_n \) are both series of positive terms, (ii) \( \sum b_n \) is convergent, and (iii) \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \), then \( \sum a_n \) is also convergent.