Midterm Exam solution
Math 311, Spring 2009

In-class part:

1. (16 points) Choose TRUE or FALSE for the following statements. If you choose FALSE, please give an example for which the statement does not hold.

   (a) T F If $A$ and $B$ are two bounded sets of real numbers, then $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
      True

   (b) T F Let $(s_n)$ be a bounded sequence of real numbers and $\limsup s_n = b < 0$. Then there exists $N$ such that when $n > N$, then $s_n$ is negative.
      True

   (c) T F If $\lim s_n$ exists, then $\lim s_{2n}$ also exists.
      True since $\lim s_{2n} = \lim s_n \cdot \lim s_n$

   (d) T F If $\lim s_{2n}$ exists, then $\lim s_n$ also exists.
      False. $s_n = (-1)^n$

   (e) T F If a series $\sum a_n^2$ is convergent, then $\sum a_n$ is also convergent.
      False. $a_n = \frac{1}{n^2}$

   (f) T F If a series $\sum a_n$ is convergent, then $\sum a_n^2$ is also convergent.
      False. $a_n = \frac{(-1)^n}{\sqrt{n}}$

   (g) T F If $\sum a_n$ is convergent, then $\limsup |a_{n+1}/a_n| < 1$.
      False. $a_n = 1/n^2$, convergent but $\lim |a_{n+1}/a_n| = 1$

   (h) T F If the sequence $\{s_n\}$ satisfies $s_n > 0$, $\lim s_n = 0$ and $(s_n)$ is decreasing, then the infinite series $\sum s_n$ is convergent.
      False. $s_n = 1/n$

2. (6 points) Let $s_n = \cos \left(\frac{\pi}{4} + \frac{n\pi}{2}\right) + (-1)^n$.
   
   (a) Find the set of subsequential limits of $(s_n)$.
   
   (b) Find $\sup\{s_n\}$ and $\inf\{s_n\}$.
   
   (c) Find a subsequence of $(s_n)$ which converges to $\sup\{s_n\}$. You need to give the selection function $\sigma(k)$ for the subsequence.
   
   (a) $\{\pm 1 \pm \sqrt{2}/2\}$; (b) $\sup\{s_n\} = 1 + \sqrt{2}/2$ and $\inf\{s_n\} = -1 - \sqrt{2}/2$; (c) $\sigma(k) = 4k$, $s_{4k} = 1 + \sqrt{2}/2$.

3. (10 points) Let $(s_n)$ be a sequence of real numbers.
   
   (a) Define what it means for $(s_n)$ to be a Cauchy sequence.
   
   (b) Give the negation of the statement “$(s_n)$ is Cauchy” according to your definition in part (a).
   
   (c) Prove if $\lim s_n = s$, then $(s_n)$ is a Cauchy sequence.
   
   (a) $(s_n)$ is a Cauchy sequence if for any $\varepsilon > 0$, there exists $N$ such that when $n, m \geq N$, then $|s_n - s_m| < \varepsilon$.
   
   (b) If $(s_n)$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$, such that for any $N$, there exist $m, n > N$ satisfying $|s_n - s_m| \geq \varepsilon$.
   
   (c) see page 60 Lemma 10.9
4. (6 points) Suppose that \((a_n)\) and \((b_n)\) are two sequences satisfying \(a_n \leq b_n\) for any \(n \in \mathbb{N}\), and \((a_n), (b_n)\) are both convergent. Prove that \(\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n\) by using the definition of the limit. (You cannot apply any theorems or exercises in textbook which directly imply this statement.)

Let \(\lim a_n = a\) and \(\lim b_n = b\). Suppose that \(a > b\), then from Archimedean property, there exists \(\varepsilon > 0\) such that \((a - b)/2 > \varepsilon > 0\). For this \(\varepsilon > 0\), from the definition of limit, there exists \(N_1\) such that when \(n > N_1\), then \(|a_n - a| < \varepsilon\), and there exists \(N_2\) such that when \(n > N_2\), then \(|b_n - b| < \varepsilon\).

We choose \(n = \max\{N_1, N_2\}\), then \(a_n - b_n < a - b\). From the choice of \(\varepsilon\), \(a - \varepsilon > b + \varepsilon\) since \((a - b)/2 > \varepsilon > 0\). So \(a_n - b_n < a - b\), which is in contradiction with the assumption \(a_n \leq b_n\) for any \(n \in \mathbb{N}\). So \(a > b\) cannot be true, and we must have \(a \leq b\).

5. (6 points) Define a sequence \((x_n)\) by \(x_1 = 2\), and \(x_{n+1} = 2 - \frac{1}{x_n}\).

(a) Prove \(x_n > 1\) for all \(n \in \mathbb{N}\).
(b) Prove \((x_n)\) is decreasing.
(c) From monotone convergent theorem, \((x_n)\) is convergent. What is the limit of \((x_n)\)?

(a) \(x_1 = 2 > 1\). If \(x_n > 1\), then \(x_{n+1} = 2 - \frac{1}{x_n} > 2 - 1 = 1\). So \(x_n > 1\) for all \(n \in \mathbb{N}\) from mathematical induction principle.
(b) \(x_1 = 2\) and \(x_2 = 1.5\) so \(x_1 > x_2\). If \(x_n > x_{n+1}\), then \(x_{n+1} - x_n = \frac{1}{x_n} - \frac{1}{x_{n+1}} = \frac{x_n - x_{n+1}}{x_n x_{n+1}} > 0\) from the assumption. Hence \(x_n > x_{n+1}\) for all \(n \in \mathbb{N}\) from mathematical induction principle.
(c) The limit \(\lim x_n = x\), and \(x\) satisfies \(x = 2 - 1/x\), or \(x^2 = 2x - 1\). Solving the equation we find \(x = 1\). So \(x = 1\).

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**Take-home part:**

1. Prove \(\sqrt{n-1} + \sqrt{n+1}\) is irrational for every \(n \in \mathbb{N}\).

Let \(x = \sqrt{n-1} + \sqrt{n+1}\). Then \(x^2 = 2n + 2\sqrt{n^2 - 1}\), so \(x\) satisfies \(x^2 - 4nx^2 + 4 = 0\). From rational zeros theorem, the only possible rational roots are \(\pm 1, \pm 2\) and \(\pm 4\), and the integer \(n\) satisfies \(n = (x^4 + 4)/(4x^2)\). One can check that for \(x = \pm 1, \pm 2\) or \(\pm 4\), none of \((x^4 + 4)/(4x^2)\) is an integer.

So for any \(n \in \mathbb{N}\), \(x^4 - 4nx^2 + 4 = 0\) has no rational roots, then \(\sqrt{n-1} + \sqrt{n+1}\) cannot be rational.

2. Using only the axioms of ordered field on page 13 to prove: if \(a \leq b\) and \(c \leq d\), then \(ad + bc \leq ac + bd\). Please state which axiom(s) you use in each step.

If \(a \leq b\), then \(0 = a - a \leq b - a\) (from O4); since \(c \leq d\) and \(0 \leq b - a\), then \(c(b - a) \leq d(b - a)\) (from O5); from (DL), \(cb - ca = c(b - a) \leq d(b - a) = db - da\); from (M2), \(bc - ac \leq bd - ad\); finally from (O4), (A4) and (A2), since \(bc - ac \leq bd - ad\), then \(ad + bc = (bc - ac) + ac + bd \leq (bd - ad) + ac + ad = bd + ac\).

3. Let \(A\) be a non-empty subset of the real numbers, which sup \(A = 3\) and inf \(A = -2\). Find the sup \(B\) and inf \(B\)

where \(B = \{\frac{1}{x} : x \in A\}\). You do not need to prove your answers by definition.

sup \(B = 25/3\) and inf \(B = 1/8 - 27 = -215/8\)

4. Prove that \(\lim_{n \to \infty} \frac{n^2 - n + 1}{2n^2 + n - 10} = \frac{1}{2}\) by using the definition of the limit of sequence. Do not use limit theorems, and explicitly show the choice of \(N\).

From calculation, we have \(\left|\frac{n^2 - n + 1}{2n^2 + n - 10} - \frac{1}{2}\right| = \frac{-3n + 12}{2(2n^2 + n - 10)}\). If \(n \geq 5\), then \(3n - 12 > 0\) and \(2n^2 + n - 10 > 0\), so \(\left|\frac{-3n + 12}{2n^2 + n - 10}\right| = \frac{3n - 12}{2n^2 + n - 10}\). Now we have \(3n - 12 \leq 3n\), and when \(n \geq 10\), we have \(2n^2 + n - 10 > 2n^2\). Thus when \(n \geq 10\), \(\frac{3n - 13}{2(2n^2 + n - 10)} < \frac{3n}{4n^2} = \frac{3}{4n}\). We shall make \(3/(4n) < \varepsilon\). For any \(\varepsilon > 0\), we choose \(N = \max\{10, 3/(4\varepsilon)\}\), then \(\left|\frac{n^2 - n + 1}{2n^2 + n - 10} - \frac{1}{2}\right| = \frac{-3n + 12}{2(2n^2 + n - 10)} < \frac{3n - 12}{2(2n^2 + n - 10)} < \frac{3}{4n} < \varepsilon\). Every step can be justified by calculation above. Hence \(\lim_{n \to \infty} \frac{n^2 - n + 1}{2n^2 + n - 10} = \frac{1}{2}\).
5. Assume that \( (a_n) \) and \( (b_n) \) are Cauchy sequences. Use triangle inequality and the definition of Cauchy sequence to show that the sequence \( (c_n) \) defined by \( c_n = |a_n - b_n| \) is also Cauchy.

From the definition Cauchy sequence, for any \( \varepsilon > 0 \), there exists \( N_1 \) such that when \( n, m > N_1 \), \( |a_n - a_m| < \varepsilon/2 \), and there exists \( N_2 \) such that when \( n, m > N_2 \), \( |b_n - b_m| < \varepsilon/2 \). Let \( N = \max\{N_1, N_2\} \).

Then when \( n, m > N \), \( |c_n - c_m| = |(a_n - b_n) - (a_m - b_m)| \leq |a_n - a_m| + |b_n - b_m| < \varepsilon \). Hence \( c_n = |a_n - b_n| \) is also Cauchy.

6. Prove that for any real number \( x \), there exists a sequence \( (x_n) \) of rational numbers, which is monotone increasing, and converges to \( x \). (Hint: compare with Exercise 10.7, and notice here we need it to be increasing not nondecreasing)

Choose a number \( x_1 < x \) (which exists from Archimedean property). Let \( S = \{ y \in \mathbb{R} : x_1 < y < x \} \), then \( S \) is nonempty from Archimedean property, and \( S \) is bounded. \( \sup S = x \) and \( x \notin S \). So all conditions in Exercise 10.7 are satisfied. The proof of Exercise 10.7 can be used to get an increasing sequence \( (s_n) \) in \( S \) so that \( \lim s_n = x \). From Archimedean property, one can select a rational number \( x_n \) so that \( s_n < x < s_{n+1} \). Then from squeeze theorem, \( \lim s_n = \lim x_n = \lim s_{n+1} = x \). This sequence \( (x_n) \) satisfies all required properties.

7. Let \( (s_n) \) be a bounded sequence of real numbers, and let \( k \) be a positive real number. Prove that \( \lim \sup (ks_n) = k \lim \sup s_n \).

Follow the proof of Theorem 12.1

8. (a) Give an example of a positive sequence \( (a_n) \) satisfying \( \lim \inf \frac{a_{n+1}}{a_n} < \lim \sup \frac{a_{n+1}}{a_n} \).

(b) Give an example of a positive sequence \( (a_n) \) satisfying

\[
\lim \inf \frac{a_{n+1}}{a_n} < \lim \inf (a_n)^{1/n} < \lim \sup (a_n)^{1/n} < \lim \sup \frac{a_{n+1}}{a_n}.
\]

For example, \( a_n = 2^{(-1)^n} \), then \( a_2n+1 = 2^{-2n+1} \) and \( a_2n = 2^{2n} \). \( a_2n+2/a_{2n+1} = 2^{4n+3} \) so \( \lim \sup \frac{a_{n+1}}{a_n} = \infty \); \( a_{2n+1}/a_{2n} = 2^{-(4n+1)} \), so \( \lim \inf \frac{a_{n+1}}{a_n} = 0 \); For \( n \) even, \( (a_n)^{1/n} = 2 \), so \( \lim \sup (a_n)^{1/n} = 2 \), and for \( n \) odd, \( (a_n)^{1/n} = 1/2 \), so \( \lim \inf (a_n)^{1/n} = 1/2 \).

9. Determine whether the following series converge. Clearly state which criterion(s) you use.

(i) \( \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2 + 2n} \); (ii) \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \); (iii) \( \sum_{n=1}^{\infty} \frac{1}{(\sqrt{3n+1} - \sqrt{3n})} \).

(i) \( \frac{\cos(n\pi)}{n^2 + 2n} \leq \frac{1}{n^2 + 2n} \leq \frac{1}{n^2} \), then it is convergent from comparison test since \( \sum (1/n^2) \) is convergent absolutely.

(ii) \( \frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} \rightarrow \frac{1}{4} \) when \( n \rightarrow \infty \), so it is convergent absolutely from ratio test.

(iii) \( \sqrt{3n+1} - \sqrt{3n} \geq \frac{1}{\sqrt{3n+1} + \sqrt{3n+1}} \geq \frac{1}{2\sqrt{3n+1}} \), but \( \sum \frac{1}{2\sqrt{3n+1}} \) is divergent, so it is divergent.

10. Use the definition of convergent series to prove that if (i) \( \sum a_n \) and \( \sum b_n \) are both series of positive terms, (ii) \( \sum b_n \) is convergent, and (iii) \( \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1 \), then \( \sum a_n \) is also convergent.

From the definition of limit, for a fixed \( \varepsilon > 0 \), there exists \( N \) such that when \( n > N \), \( |\frac{a_n}{b_n} - 1| < \varepsilon \), which implies \( (1-\varepsilon)b_n \leq a_n \leq (1+\varepsilon)b_n \). Define \( c_n = a_n \) for \( n \leq N \), and \( c_n = (1+\varepsilon)b_n \) for \( n > N \). Then \( \sum c_n \) is convergent since \( \sum c_n = \sum_{n=1}^{N} a_n + (1+\varepsilon) \sum_{n=N+1}^{\infty} b_n \). The convergence of \( \sum b_n \) implies the convergence of \( \sum c_n \). Finally since \( a_n \leq c_n \) for all \( n \), then \( \sum a_n \) is convergent from the convergence of \( \sum c_n \) and the comparison test.