Eigenvalues and Eigenfunctions of the Laplacian on Isotropic Quantum Graphs

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Abstract

We present solutions of the Laplacian eigenvalue problem on several simple quantum graphs. Three basic graph structures are defined; we primarily consider the case when all edges of the graph in question are of the same length, or isotropic. Isotropic graphs provide a symmetry which can be exploited to calculate the solutions. Eigenvalues and eigenfunctions for these isotropic structures are characterized; anisotropic structures are also examined. Additional properties of eigenvalues and eigenfunctions on these graphs are demonstrated.

Preliminaries

The eigenvalue problem is an elementary problem within the field of partial differential equations. One of the simplest and most applicable Hermitian operators is the Laplacian, alternatively noted by Δ or ∇^2. In one dimension, the Laplacian is simply the second derivative [1]:

$$\Delta = \frac{\partial^2}{\partial x^2} \tag{1}$$

The Laplacian is most commonly encountered during the process of separation of variables for the Diffusion or Wave equations, given below [1]:

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \tag{2}$$

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2} \tag{3}$$

These are actually examples of simple parabolic and hyperbolic PDEs. Once separation of variables has been applied, the problem is generally stated as

$$\Delta u = -\mu u \tag{4}$$

However, the actual problem still requires specification of the domain and boundary conditions. In a common physical application, the domain is chosen to be the shape of the waveguide or diffusion chamber, and is commonly two- or three- dimensional [1]. The quantum graph, however, instead defines the problem on a collection of individual one-dimensional domains. Individually, these are simple; however, the quantum graph (through the definition of boundary conditions) turns the trivial problem into a more interesting coupled problem.

The vertices of the quantum graph require careful characterization. These points are where interaction with the rest of the graph occur. On free end points, one is free to choose boundary conditions, such as the elementary Dirichlet or Neumann conditions. (In our treatment, we restrict our calculations to Neumann end point conditions.) However, on junctions, a new type of condition is imposed, unique to quantum graphs. These conditions are often referred to as Kirchoff conditions, and can be stated as:

$$\Sigma\frac{\partial u_i}{\partial x} = 0 \tag{5}$$

Basic Graphs

We made several restrictions on the types of graphs which we studied. Our primary assumption was that the lengths of the graph edges were all identical (hence "isotropy;" any edge you choose has the same length). Another assumption was that self-looping was allowed; that is, a vertex may have an edge connected to itself. With these assumptions in mind, several basic graph classes were natural to characterize.

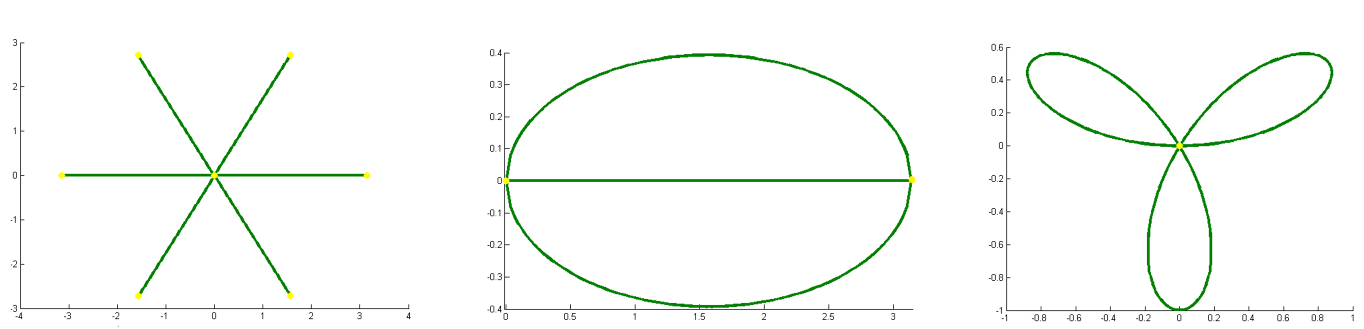

Figure: Basic Graph Classes.

Elementary Solutions

Our elementary solution comes from the most basic domain, the one-dimensional line. For Neumann boundary conditions, the solutions are well known. The solutions are described by

$$\mu_n = \frac{n^2\pi^2}{L^2} \quad , \quad u_n(x) = \cos\left(\frac{n\pi x}{L}\right) \tag{6}$$

To solve more complicated graph domains, one combines the end boundary conditions with the joint boundary conditions and the continuity requirement. Because the solutions on the subdomains are part of the complete solution, and the joint vertices are points common to more than one subdomain, then continuity of the solution must apply. Taken together, the collection of boundary conditions ensures that a solution can be found.

The n-Star

In general, the solutions of the n-star can be grouped into two classes of solutions. The first class has eigenvalues and eigenfunctions of the form

$$\mu_n = \frac{n^2\pi^2}{L^2} \quad , \quad u_n(x) = a_0\begin{pmatrix}1\\1\\1\\\cdots\\1\end{pmatrix}\cos\left(\frac{n\pi x}{L}\right) \tag{7}$$

The other class of solutions has eigenvalues and eigenfunctions of the form

$$\mu_m = \frac{(m+\frac{1}{2})^2\pi^2}{L^2} \quad , \quad u_m(x) = \left(a_1\begin{pmatrix}1\\-1\\0\\\cdots\\0\end{pmatrix} + \ldots + a_n\begin{pmatrix}1\\0\\\cdots\\0\\-1\end{pmatrix}\right)\sin\left(\frac{(m+\frac{1}{2})\pi x}{L}\right) \tag{8}$$

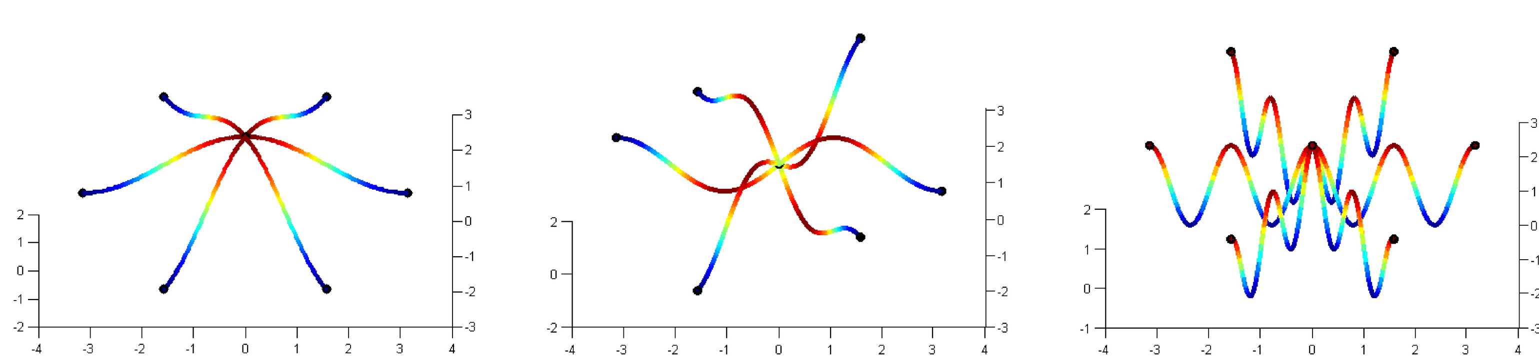

Figure: Some solutions of the 6-Star.

The n-Path

The n-path has, in general, only one class of eigenvalues and eigenfunctions, which are of the form

$$\mu_n = \frac{n^2\pi^2}{L^2} \quad , \quad u_n(x) = a_0\begin{pmatrix}1\\1\\1\\\cdots\\1\end{pmatrix}\cos\left(\frac{n\pi x}{L}\right) + \left(a_1\begin{pmatrix}1\\-1\\0\\\cdots\\0\end{pmatrix} + \ldots + a_n\begin{pmatrix}1\\0\\\cdots\\0\\-1\end{pmatrix}\right)\sin\left(\frac{n\pi x}{L}\right) \tag{9}$$

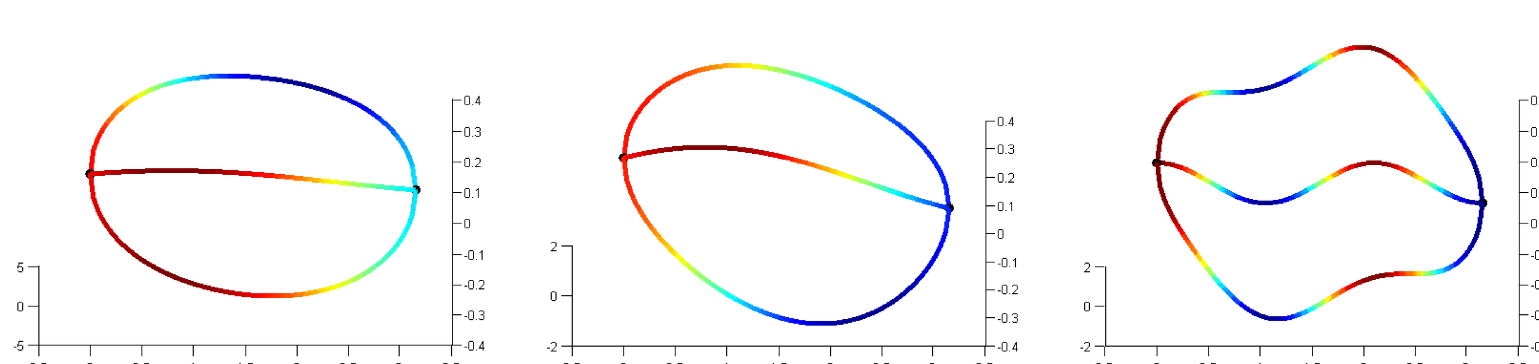

Figure: Some solutions of the 3-Path.

The n-Petal

The n-petal has only even eigenvalues, with solutions

$$\mu_n = \frac{(2n)^2\pi^2}{L^2} \quad , \quad u_n(x) = a_0\begin{pmatrix}1\\1\\1\\\cdots\\1\end{pmatrix}\cos\left(\frac{2n\pi x}{L}\right) + \left(a_1\begin{pmatrix}1\\-1\\0\\\cdots\\0\end{pmatrix} + \ldots + a_n\begin{pmatrix}1\\0\\\cdots\\0\\-1\end{pmatrix}\right)\sin\left(\frac{2n\pi x}{L}\right) \tag{10}$$

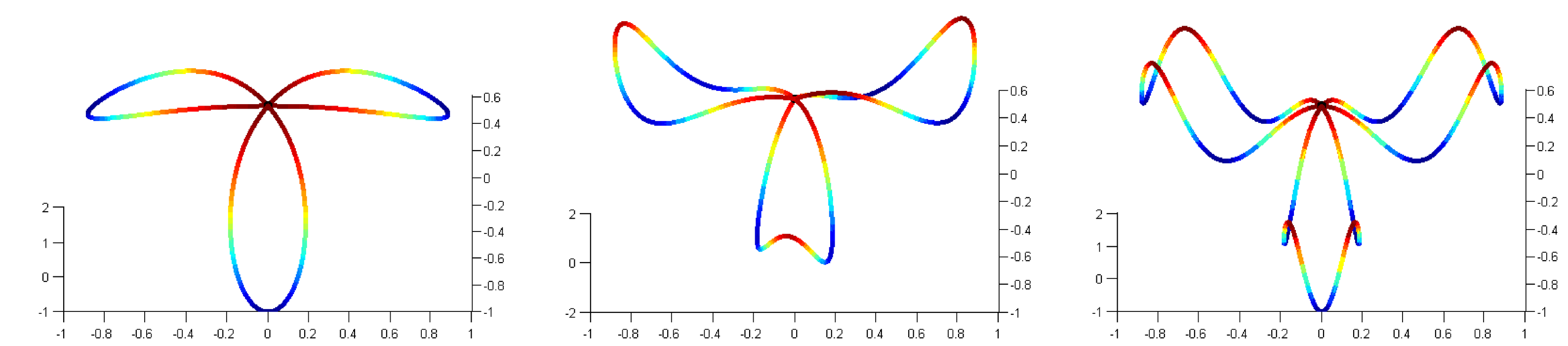

Figure: Some solutions of the 3-Petal.

Future Work

A logical next step would be to consider relaxing the requirement of isotropy. This is a natural application of perturbation theory. However, rather than perturbing the solutions themselves, the lengths of the edges would require perturbation. Within the formulation of the problem, this would be accomplished by perturbing the form of the problem itself. (What is perturbed physically depends on the problem in question. If you consider the problem from the perspective of diffusion, then the diffusion coefficient D is perturbed; from the perspective of waves, the wave speed c is perturbed.)

Another line of inquiry involves exploring the mathematical relationship between the classes of graphs themselves, and the solutions on them. In a way, you can look at the n- path as the n-petal whose central vertex was split into two vertices. Establishing a relationship between them could lead to an inductive relation between another class of graphs, such as an "n-chain."

References

[1] Logan, J. David, *Applied Mathematics, Third Edition*, John Wiley & Sons, Inc., 2006.

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Acknowledgments

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