JOURNAL OF FUNCTIONAL ANALYSIS 7, 487–513 (1971)

Some Global Results for Nonlinear Eigenvalue Problems

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Communicated by Peter D. Lax

Received May 28, 1970

In this paper we investigate the structure of the solution set for a large class of nonlinear eigenvalue problems in a Banach space. Our main results demonstrate the existence of continua, i.e., closed connected sets, of solutions of these equations. Although the emphasis is on the case when bifurcation occurs, the nonbifurcation situation is also treated. Applications are given to ordinary and partial differential equations and to integral equations.

INTRODUCTION

In this paper we investigate the structure of the solution set for a large class of nonlinear eigenvalue problems in a Banach space. Our main results demonstrate the existence of continua, i.e., closed connected sets, of solutions of these equations. Although the emphasis is on the case when bifurcation occurs, the nonbifurcation situation is also treated. Applications are given to ordinary and partial differential equations and to integral equations.

Let $\mathscr{F} : \mathscr{E} \to \mathscr{E}_1$ where $\mathscr{E}$ and $\mathscr{E}_1$ are real Banach spaces and $\mathscr{F}$ is continuous. Suppose the equation $\mathscr{F}(U) = 0$ possesses a simple curve of solutions $\mathscr{C}$ given by $\{U(t) \mid t \in [a, b]\}$. If for some $\tau \in (a, b)$, $\mathscr{F}$ possesses zeroes not lying on $\mathscr{C}$ in every neighborhood of $U(\tau)$, then $U(\tau)$ is said to be a *bifurcation point* for $\mathscr{F}$ with respect to the curve $\mathscr{C}$.

A special family of such equations has the form

$$u = G(\lambda, u) \tag{0.1}$$

where $\lambda \in \mathbf{R}$, $u \in E$, a real Banach space with norm $\|\cdot\|$ and $G : \mathscr{E} \equiv \mathbf{R} \times E \to E$ is compact and continuous. In addition, $G(\lambda, u) = \lambda L u + H(\lambda, u)$, where $H(\lambda, u)$ is $0(\| u \|)$ for u near 0 uniformly on bounded λ intervals and L is a compact linear map on E. A solution of (0.1) is a pair $(\lambda, u) \in \mathscr{E}$. The known curve of solutions $\{(\lambda, 0 \mid \lambda \in \mathbf{R}\}$

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will henceforth be referred to as the *trivial solutions*. The closure of the set of nontrivial solutions of (0.1) will be denoted by $\mathscr{S}$.

Equations of the form (0.1) are usually called *nonlinear eigenvalue problems* and arise in many contexts in mathematical physics. It is therefore of interest to investigate the structure of the set of their solutions. Let $r(L)$ denote the set of $\mu \in \mathbf{R}$ such that there exists $v \in E$, $v \neq 0$, with $v = \mu L v$, i.e., $r(L)$ consists of the reciprocals of the real nonzero eigenvalues of L. Following [1], we call $\mu \in r(L)$ a characteristic value of L.

It is well-known that the possible bifurcations points for (0.1) with respect to the curve of trivial solutions lie in $\{(\mu, 0) | \mu \in r(L)\}$ [1]. In fact if $\mu \in r(L)$ is of odd multiplicity, $(\mu, 0)$ is a bifurcation point. We will show much more, namely that there exists a continuum of solutions of (0.1) in $\mathscr{S}$ which meets $(\mu, 0)$ and either meets ∞ in $\mathscr{E}$ or meets $(\hat{\mu}, 0)$, where $\mu \neq \hat{\mu} \in r(L)$. Therefore, bifurcation from characteristic values of odd multiplicity is a global rather than a local phenomenon. Partial results in this direction already appear in [1]. We will describe them more fully later.

The proof of the above theorem as well as related results will be carried out in Section 1. Examples are given showing that both possibilities of the theorem may occur. Additional results are obtained if μ is a real simple characteristic value of L. For this case a pair of continua in $\mathscr{S}$ can be associated with $(\mu, 0)$. The main tool required for the proofs of these results is the theory of degree of mapping of Leray–Schauder [1–3, Appendix].

Various applications of the results will be given in Sections 2. In particular, some problems for second order ordinary differential equations and integral equations are treated in which nodal properties for solutions implies that the various continua meet ∞ in $\mathscr{E}$. The question of the existence of positive solutions to quasilinear elliptic partial differential equations is also considered.

Lastly in Section 3 it will be shown how some of the ideas used in Section 1 can be employed to prove the existence of global continua of solutions for problems which need not be a bifurcative nature. We treat equations of the form

$$u = T(\lambda, u), \tag{0.2}$$

where again $T : \mathscr{E} \to E$ is compact and continuous but $T(\lambda, 0)$ need not equal zero. Here $(0, 0)$ is a solution and, as we shall show, lies on a pair of continua of solutions of (0.2) meeting ∞ in $\mathscr{E}$. Applications will be given to quasilinear elliptic partial differential equations and also nonlinear wave equations.

Many people have worked on nonlinear eigenvalue problems; in particular, for ordinary differential equations and integral equations. See [1, 4 and 5] where several references are given. This paper was motivated by our earlier results in [3] and many of the ideas used here already appear in that paper within its special context.

1. The Odd Multiplicity Results

We begin with some definitions and technical lemmas. Let $\mathscr{E}$ and G be as in the Introduction. For $\mathcal{O} \subset \mathscr{E}$, $\partial\mathcal{O}$ denotes the boundary of $\mathcal{O}$. By a subcontinuum of $\mathcal{O}$ we mean a subset of $\mathcal{O}$ which is closed and connected in $\mathscr{E}$. We say a continuum $\mathscr{C}$ of $\mathscr{E}$ meets infinity if $\mathscr{C}$ is not bounded. A useful result on continua is [6]:

LEMMA 1.1. *Let K be a compact metric space and A and B disjoint closed subsets of K. Then either there exists a subcontinuum of K meeting both A and B or $K = K_A \cup K_B$, where K_A, K_B are disjoint compact subsets of K containing A and B, respectively.*

As norm in $\mathscr{E}$, we take $\|(\lambda, u)\| = (|\lambda|^2 + \|u\|^2)^{1/2}$. Let $\mathscr{B}_\epsilon$ and B_ϵ denote respectively open balls in $\mathscr{E}$ and E of radius ϵ centered at $(\mu, 0)$, 0.

LEMMA 1.2. *Let $\mu \in r(L)$. Suppose that there does not exist a subcontinuum of $\mathscr{S} \cup \{(\mu, 0)\}$ which meets $(\mu, 0)$ and either*

(i) *meets infinity in $\mathscr{E}$, or*

(ii) *meets $(\hat{\mu}, 0)$, where $\mu \neq \hat{\mu} \in r(L)$.*

Then there exists a bounded open set $\mathcal{O} \subset \mathscr{E}$ such that $(\mu, 0) \in \mathcal{O}$, $\partial\mathcal{O} \cap \mathscr{S} = \varnothing$, and $\mathcal{O}$ contains no trivial solutions other than those in $\mathscr{B}_\epsilon$ where $0 < \epsilon < \epsilon_0$, ϵ_0 being the distance from μ to $(r(L) - \{\mu\})$.

Proof. Let $\mathscr{C}_\mu$ denote the (connected) component, i.e., the maximal connected subset, of $\mathscr{S} \cup \{(\mu, 0)\}$ to which $(\mu, 0)$ belongs. By (i), this is a bounded subset of $\mathscr{E}$ and, therefore, by the continuity and compactness of G, is compact. Let u_δ be a δ-neighborhood of $\mathscr{C}_\mu$. For $\delta < \epsilon_0$ sufficiently small, by (ii) together with the fact that $(\lambda, 0)$ is an isolated solution of (0.1) if $\lambda \notin r(L)$, we can assume $\mathscr{U}_\delta$ contains no solutions $(\lambda, 0)$ of (0.1) for $|\lambda - \mu| > \delta$.

Let $K \equiv \overline{\mathscr{U}}_\delta \cap \mathscr{S}$. Then since $\mathscr{S}$ is locally compact in $\mathscr{E}$, K is a compact metric space under the induced topology from $\mathscr{E}$ and

$\mathscr{C}_\mu \cap \partial u_\delta = \varnothing$ by construction. By Lemma 1.1, there exist disjoint compact subsets A, B of K such that $\mathscr{C}_\mu \subset A$, $(\partial u_\delta) \cap \mathscr{S} \subset B$ and $K = A \cup B$. Let $\mathcal{O}$ be any ϵ neighborhood in $\mathscr{E}$ of A where ϵ is less than the distance between A and B. Then $\mathcal{O}$ satisfies the requirements of the lemma.

Suppose $\mu \in r(L)$. The multiplicity of μ is the dimension of $\bigcup_{j=1}^{\infty} N((\mu L - I)^j)$ where I is the identity map on E and $N(P)$ denotes the null space of P. Since L is compact, μ is of finite multiplicity.

Let $\Omega \subset E$ be bounded and open, $\Psi(u) = u - T(u)$ where T is continuous and compact on $\bar{\Omega}$, and $b \in E$, $b \notin \Psi(\partial\Omega)$. Then the Leray–Schauder degree of Ψ with respect to Ω and b is well defined and will be denoted by $d(\Psi, \Omega, b)$. [1–3, Appendix]. In what follows, $b = 0$, and therefore we just write $d(\Psi, \Omega)$. The index of an isolated zero u_0 of Ψ will be denoted by $i(\Psi, u_0)$.

Next, let $\Phi(\lambda, u) = u - G(\lambda, u)$. When the u dependence of Φ is not important, we just write $\Phi(\lambda)$. For fixed λ, $\Phi(\lambda)$ is of the appropriate form for the use of Leray–Schauder degree. With the aid of d and Lemma 1.2, we can prove our first global result.

THEOREM 1.3. *If $\mu \in r(L)$ is of odd multiplicity, then $\mathscr{S}$ possesses a maximal subcontinuum $\mathscr{C}_\mu$ such that $(\mu, 0) \in \mathscr{C}_\mu$ and $\mathscr{C}_\mu$ either*

(i) *meets infinity in $\mathscr{E}$, or*

(ii) *meets $(\hat{\mu}, 0)$, where $\mu \neq \hat{\mu} \in r(L)$.*

Proof. By a maximal $\mathscr{C}_\mu$ we mean $\mathscr{C}_\mu$ is not a proper subcontinuum of any $\mathscr{C} \subset \mathscr{S}$ having the above properties.

If there does not exist $\mathscr{C}_\mu$ as above, there exists $\mathcal{O}$ and δ as in Lemma 1.2. Let $\mathcal{O}_\lambda = \{u \in E \mid (\lambda, u) \in \mathcal{O}\}$. For $0 < |\lambda - \mu| \leqslant \delta$, $(\lambda, 0)$ is an isolated solution of (0.1). Therefore there exists $\rho(\lambda) > 0$ such that $(\lambda, 0)$ is the only solution of (0.1) in $\{\lambda\} \times \bar{B}_{\rho(\lambda)}$. Let $\rho(\lambda) = \rho(\mu + \delta)$ for $\lambda > \mu + \delta$ and $\rho(\lambda) = \rho(\mu - \delta)$ for $\lambda < \mu - \delta$. By choosing $\rho(\mu \pm \delta)$ small enough, it can be assumed that $\bar{B}_{\rho(\lambda)} \cap \bar{\mathcal{O}}_\lambda = \varnothing$ if $|\lambda - \mu| \geqslant \delta$. For $\lambda \neq \mu$ there are no solutions of (0.1) on $\{\lambda\} \times \partial(\mathcal{O}_\lambda - \bar{B}_{\rho(\lambda)})$ and therefore $d(\Phi(\lambda), \mathcal{O}_\lambda - \bar{B}_{\rho(\lambda)})$ is well-defined. We will show first that

$$d(\Phi(\lambda), \mathcal{O}_\lambda - \bar{B}_{\rho(\lambda)}) = 0 \qquad \lambda \neq \mu, \tag{1.4}$$

and then that it is not possible for Eq. (1.4) to hold for all λ near μ. With the aid of this contradiction, the theorem is established.

Let $\lambda > \mu$. Choose $\lambda^* > \lambda$ so large that $(\nu, u) \in \mathcal{O}$ implies that $\nu < \lambda^*$. Let $\rho = \inf\{\rho(\theta) \mid \lambda \leqslant \theta \leqslant \lambda^*\}$. It is easily seen that $\rho > 0$. Then $\mathscr{U} = \mathcal{O} - [\lambda, \lambda^*] \times \bar{B}_\rho$ is a bounded open set in $\hat{\mathscr{E}} \equiv [\lambda, \lambda^*] \times E$

and $\Phi(\theta, u) \neq 0$ for $(\theta, u) \in \partial\mathscr{U}$ (in $\hat{\mathscr{E}}$). Therefore by the homotopy invariance of d, [2 and 3, Appendix],

$$d(\Phi(\theta), \mathcal{O}_\theta - \bar{B}_\rho) \equiv \text{constant}, \qquad \theta \in [\lambda, \lambda^*]. \tag{1.5}$$

Since $\mathcal{O}_{\lambda^*} - \bar{B}_\rho = \varnothing$,

$$d(\Phi(\lambda^*), \mathcal{O}_{\lambda^*} - \bar{B}_\rho) = 0. \tag{1.6}$$

Thus Eqs. (1.5) and (1.6) imply

$$d(\Phi(\lambda), \mathcal{O}_\lambda - \bar{B}_\rho) = 0. \tag{1.7}$$

Since $\Phi(\lambda)$ has no zeroes in $\{\lambda\} \times (B_\rho - \bar{B}_{\rho(\lambda)})$, Eq. (1.7) and the additivity of d imply Eq. (1.4) for $\lambda > \mu$. If $\lambda < \mu$ a similar argument is employed.

Again using the homotopy invariance of d,

$$d(\Phi(\lambda), \mathcal{O}_\lambda) \equiv \text{constant} \qquad | \lambda - \mu | < \epsilon. \tag{1.8}$$

Let $\mu - \epsilon < \underline{\lambda} < \mu < \bar{\lambda} < \mu + \epsilon$. By the additivity of d and the fact that $(\lambda, 0)$ is an isolated zero of $\Phi(\lambda)$ for $\lambda \notin r(L)$,

$$\begin{aligned} d(\Phi(\underline{\lambda}), \mathcal{O}_{\underline{\lambda}}) &= i(\Phi(\underline{\lambda}), (\underline{\lambda}, 0)) + d(\Phi(\underline{\lambda}), \mathcal{O}_{\underline{\lambda}} - \bar{B}_{\rho(\underline{\lambda})}), \\ d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}}) &= i(\Phi(\bar{\lambda}), (\bar{\lambda}, 0)) + d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}} - \bar{B}_{\rho(\bar{\lambda})}). \end{aligned} \tag{1.9}$$

Combining Eqs. (1.4) and (1.9) gives

$$\begin{aligned} d(\Phi(\underline{\lambda}), \mathcal{O}_{\underline{\lambda}}) &= i(\Phi(\underline{\lambda}), (\underline{\lambda}, 0)). \\ d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}}) &= i(\Phi(\bar{\lambda}), (\bar{\lambda}, 0)). \end{aligned} \tag{1.10}$$

So by (1.8),

$$i(\Phi(\underline{\lambda}), (\underline{\lambda}, 0) = i(\Phi(\bar{\lambda}), (\bar{\lambda}, 0)). \tag{1.11}$$

However since μ is a characteristic value of L of odd multiplicity, $i(\Phi(\underline{\lambda}), (\underline{\lambda}, 0) = -i(\Phi(\bar{\lambda}), (\bar{\lambda}, 0)) \neq 0$. Thus we have a contradiction and the proof is complete.

If G is not globally defined, a somewhat weaker result prevails.

COROLLARY 1.12. *If Ω is a bounded open set in $\mathscr{E}$ containing $(\mu, 0)$, and $G(\lambda, u)$ is continuous and compact on $\bar{\Omega}$, and $\mu \in r(L)$ is of odd multiplicity, then $\mathscr{S}$ possesses a maximal subcontinuum $\mathscr{C}_\mu \subset \bar{\Omega}$ such that $(\mu, 0) \in \mathscr{C}_\mu$ and $\mathscr{C}_\mu$ either*

(i) *meets $\partial\Omega$, or*

(ii) *meets $(\hat{\mu}, 0)$, where $\mu \neq \hat{\mu} \in r(L)$, $(\hat{\mu}, 0) \in \Omega$.*

Proof. The proof is essentially the same as that of Lemma 1.2 and Theorem 1.3 and will be omitted.

If $\mu \in r(L)$ is of even multiplicity, simple examples show that $(\mu, 0)$ need not be a bifurcation point for Eq. (0.1). Thus no analog of Theorem 1.3 is possible for this case without further conditions on G.

It is of interest to compare Theorem 1.3 to previous results of Krasnoselski [1]. For Eq. (0.1) Krasnoselski calls λ a *characteristic value* and u an eigenvector, if (in our terminology) (λ, u) is a nontrivial solution of Eq. (0.1). The set of characteristic values of G is called the *spectrum* of G. The set of eigenvectors is said to form a *continuous branch* passing through an eigenvector u_0 if every bounded open set in E containing u_0 and of small diameter possesses an eigenvector on its boundary. It is proved in [1] that if $\mu \in r(L)$ is a characteristic value of odd multiplicity, then $(\mu, 0)$ is a bifurcation point for Eq. (0.1) corresponding to a continuous branch of eigenvectors passing through $u_0 = 0$. Moreover if some bounded open set in E containing 0 has no solutions on its boundary, then the spectrum of G is continuous, i.e., contains an interval near μ. If μ, as above, is an isolated point in the spectrum of $\mathcal{O}$, then the set of eigenvectors corresponding to μ forms a continuous branch intersecting every bounded open set in E containing 0.

By projecting $\mathscr{C}_\mu$ on $\mathbf{R}$ or E, it is easily seen that the above results are a consequence of Theorem 1.3. In fact, we see if μ is not an isolated point in the spectrum of G, the spectrum is continuous.

Next we will illustrate how both alternatives of Theorem 1.3 are possible. The simplest example of (i) is the linear case $H \equiv 0$. Examples of (ii) are more complicated. A sufficient condition for (ii) to occur is that $\mathscr{C}_\mu$ be bounded. We give an example of this nature due to M.G. Crandall and the author. Let $E = \mathbf{R}^2$. If $u = (u_1, u_2) \in E$, $\| u \| = (u_1^2 + u_2^2)^{1/2}$. Consider the equation

$$Au = \lambda(u - B(u)\, u), \tag{1.13}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad B(u) = \begin{pmatrix} 4u_1^2 + 6u_2^2 & -2u_1u_2 \\ -2u_1u_2 & 6u_1^2 + 4u_2^2 \end{pmatrix}$$

By inverting A, Eq. (1.13) may be put in the form (0.1) with $L = A^{-1}$ which has characteristic values $\frac{1}{2}$ and 1. Taking the inner product of Eq. (1.13) with u leads to the estimate

$$(u_1^2 + u_2^2)^2 \leqslant (B(u)\, u, u) \leqslant u_1^2 + u_2^2. \tag{1.14}$$

This implies $\| u \| \leqslant 1$ for all solutions (λ, u) of Eq. (1.13). Therefore, the projection of $\mathscr{C}_{1/2}$ and $\mathscr{C}_1$ on E is bounded. If the first alternative of Theorem 1.3 were to hold, there would exist a sequence (λ_n, u_n) of solutions of (1.13) with $\lambda_n \geqslant n$. Dividing Eq. (1.13) by λ_n, letting $n \to \infty$, and using the bounds for $\| u_n \|$, a subsequence of the u_n can be found which converges to a solution v of the "limit equation"

$$B(u)\, u = u. \tag{1.15}$$

Moreover, $v \neq 0$ for otherwise dividing Eq. (1.13) by $\| u_n \|$ and letting $n \to \infty$ gives a contradiction. The matrix $B(u)$ can be written as $B(u) = T^{-1}(\theta)\, D(r)\, T(\theta)$, where $u_1 = r \cos \theta$, $u_2 = r \sin \theta$ and

$$D(r) = \begin{pmatrix} 4r^2 & 0 \\ 0 & 6r^2 \end{pmatrix} \quad \text{and} \quad T(\theta) = \begin{pmatrix} \cos\left(\theta + \frac{\pi}{4}\right) & \sin\left(\theta + \frac{\pi}{4}\right) \\ -\sin\left(\theta + \frac{\pi}{4}\right) & \cos\left(\theta + \frac{\pi}{4}\right) \end{pmatrix}.$$

Rewriting (1.15) as $D(r)\, T(\theta) u = T(\theta) u$ leads to $1 = 4r^2 = 6r^2$ which is not possible. Thus since alternative (i) of Theorem 1.3 cannot occur here, (ii) must.

It would be of interest to find general conditions on G which imply one or the other of the alternatives of Theorem 1.3.

Remark. In a recent paper [7], a study was made of a pair of interlocking nonlinear ordinary differential equations arising from a problem in the buckling of spherical shells. This problem can be put in the form (0.1). All of the characteristic values of the L occuring there are simple. Moreover, it was shown numerically that for all cases considered (ii) occured, i.e., the solution branches always intersected.

Remark. Suppose the solutions of Eq. (0.1) are a priori bounded in the sense that there exists a continuous function $M : [0, \infty) \to [0, \infty]$ such that if $(\lambda, u) \in \mathscr{S}$ and $\lambda \geqslant 0$, $\| u \| \leqslant M(\lambda)$. Then if $\mu > 0$ is as in Theorem 1.3, $\mathscr{C}_\mu$ cannot meet ∞ and have a bounded projection on $\mathbf{R}$. Such a priori bounds occur in many problems. For example, they often occur in buckling problems in elasticity and in problems involving rotating or convecting fluids (see [5]).

Next we will prove a result about the general odd multiplicity case when the first possibility of Theorem 1.3 does not occur as, e.g., in the above matrix example.

THEOREM 1.16. *Suppose the hypotheses of Theorem* 1.3 *are satisfied and* (i) *does not occur. Let* $\Gamma = \{\gamma \in r(L) \mid (\gamma, 0) \in \mathscr{C}_\mu, \gamma \neq \mu\}$. *Then* Γ *contains at least one characteristic value of odd multiplicity.*

Proof. Since (i) does not hold, Γ contains finitely many points which we order by size: $\gamma_1 < \cdots < \gamma_j$. Arguing as in Lemma 1.2, a bounded open set $\mathcal{O} \subset \mathscr{E}$ can be found such that $\mathcal{O} \supset \mathscr{C}_\mu$, $\partial\mathcal{O} \cap \mathscr{S} = \varnothing$, and $\mathcal{O}$ contains no trivial solutions other than those within ϵ of μ or some $\gamma \in \Gamma$, where $\epsilon < \epsilon_1$, the distance from $\Gamma \cup \{\mu\}$ to the rest of $r(L)$. We define $\mathcal{O}_\lambda$ as in Theorem 1.3 and likewise $\rho(\lambda)$ which can again be taken to be constant outside of ϵ-neighborhoods of μ and the γ_r, $1 \leqslant r \leqslant j$.

Suppose $\gamma_1, \ldots, \gamma_j$ are all of even multiplicity. The computations of Theorem 1.3 and in particular Eqs. (1.8) and (1.9) show that

$$\begin{aligned} &d(\Phi(\underline{\lambda}), \mathcal{O}_{\underline{\lambda}} - \bar{B}_{\rho(\underline{\lambda})}) + i(\Phi(\underline{\lambda}), (\underline{\lambda}, 0)) \\ &\quad = d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}} - \bar{B}_{\rho(\bar{\lambda})}) + i(\Phi(\bar{\lambda}), (\bar{\lambda}, 0)). \end{aligned} \tag{1.17}$$

Since $i(\Phi(\bar{\lambda}), (\bar{\lambda}, 0)) = -i(\Phi(\underline{\lambda}), (\underline{\lambda}, 0)) \neq 0$, at least one of the integers $d(\Phi(\underline{\lambda}), \mathcal{O}_{\underline{\lambda}} - \bar{B}_{\rho(\underline{\lambda})})$, $d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}} - \bar{B}_{\rho(\bar{\lambda})})$ is nonzero.

Let γ_s be the smallest member of Γ which is larger than μ. For $\lambda \in (\mu, \gamma_s)$, an argument as in Theorem 1.3 yields

$$d(\Phi(\lambda), \mathcal{O}_\lambda - \bar{B}_{\rho(\lambda)}) = d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}} - \bar{B}_{\rho(\bar{\lambda})}). \tag{1.18}$$

As in (1.8),

$$d(\Phi(\lambda), \mathcal{O}_\lambda) \equiv \text{constant} \qquad |\lambda - \gamma_s| < \epsilon. \tag{1.19}$$

Let $\gamma_s - \epsilon < \underline{\gamma} < \gamma_s < \bar{\gamma} < \gamma_s + \epsilon$. Since γ_s is of even multiplicity, $i(\Phi(\underline{\gamma}), (\underline{\gamma}, 0)) = i(\Phi(\bar{\gamma}), (\bar{\gamma}, 0))$, and the argument of Eqs. (1.8) and (1.9) gives

$$d(\Phi(\underline{\gamma}), \mathcal{O}_{\underline{\gamma}} - \bar{B}_{\rho(\underline{\gamma})}) = d(\Phi(\bar{\gamma}), \mathcal{O}_{\bar{\gamma}} - \bar{B}_{\rho(\bar{\gamma})}). \tag{1.20}$$

Combining Eqs. (1.18) and (1.20) and using the properties of d yields

$$d(\Phi(\lambda), \mathcal{O}_\lambda - \bar{B}_{\rho(\lambda)}) = d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}} - \bar{B}_{\rho(\bar{\lambda})}), \qquad \lambda \in (\gamma_s, \gamma_{s+1}). \tag{1.21}$$

Continuing this argument and noting that $\mathcal{O}_\lambda = \varnothing$ for λ sufficiently larger than γ_j, we find that

$$d(\Phi(\bar{\lambda}), \mathcal{O}_{\bar{\lambda}} - \bar{B}_{\rho(\bar{\lambda})}) = 0. \tag{1.22}$$

A similar argument for $\lambda < \mu$ implies that

$$d(\Phi(\underline{\lambda}), \mathcal{O}_{\underline{\lambda}} - \bar{B}_{\rho(\underline{\lambda})}) = 0. \tag{1.23}$$

But Eqs. (1.22) and (1.23) contradict (1.17). Thus the result is established.

The compactness of L implies that its characteristic values are of finite multiplicity. In general, however, it is a very difficult question to determine the multiplicity of a characteristic value of L. In many applications, the characteristic value (or values) of interest are simple, i.e., of multiplicity 1 (see [5] and section 2.). More can be said about $\mathscr{C}_\mu$ for this case. In particular, if $H(\lambda, u)$ is Fréchet differentiable near $(\mu, 0)$, then $\mathscr{C}_\mu$ near $(\mu, 0)$ is given by a curve $(\lambda(\alpha), u(\alpha)) = (\mu + 0(1), \alpha v + 0(|\alpha|))$ for α near 0 where v is an eigenvector corresponding to μ. Note that we can distinguish between two portions of this curve, namely those parametrized by $\alpha \geqslant 0$ and by $\alpha \leqslant 0$. We shall show that for the general simple characteristic value case, $\mathscr{C}_\mu$ can be decomposed into two subcontinua which near $(\mu, 0)$ have only $(\mu, 0)$ as a common point.

First some preliminaries. Let μ be a simple characteristic value of L and let $v \in E$, $\ell \in E'$ (the dual of E) be corresponding eigenvectors of L and L^T, the transpose of L, normalized so that $\| v \| = 1$ and $\langle \ell, v\rangle = 1$, where $\langle \cdot, \cdot \rangle$ denotes the duality between E and E'. Let $E_1 = \{u \in E \mid \langle \ell, u\rangle = 0\}$. Then $E = \mathbf{R} \oplus E_1$ with $u = \alpha v + w$, where $\alpha = \langle \ell, u\rangle$ and $w \in E_1$.

For $\xi, \eta \in \mathbf{R}$ where e.g. $0 < \xi$, $0 < \eta < 1$, define

$$K_{\xi,\eta} = \{(\lambda, u) \in \mathscr{E} \mid |\lambda - \mu| < \xi, |\langle \ell, u\rangle| > \eta \| u \|\}.$$

Then $K_{\xi,\eta}$ is an open subset of $\mathscr{E}$ and consists of two disjoint convex components $K^+_{\xi,\eta}$, $K^-_{\xi,\eta}$, where $(\lambda, u) \in K^+_{\xi,\eta}$ implies $\langle \ell, u\rangle > \eta \| u \|$ and $(\lambda, u) \in K^-_{\xi,\eta}$ implies $\langle \ell, u\rangle < -\eta \| u \|$.

LEMMA 1.24. *There exists a $\zeta_0 > 0$ such that for all $\zeta < \zeta_0$ $(\mathscr{S} - \{(\mu, 0)\}) \cap \mathscr{B}_\zeta \subset K_{\xi,\eta}$. If $(\lambda, u) \in (\mathscr{S} - \{(\mu, 0)\}) \cap \mathscr{B}_\zeta$, then $u = \alpha v + w$, where $|\alpha| > \eta \| u \|$. Moreover, $|\lambda - \mu| = o(1)$, $w = o(|\alpha|)$ for α near 0.*

Proof. If there is no ζ_0 as above, there exist sequences $\zeta_n \searrow 0$ and $(\lambda_n, u_n) \in (\mathscr{S} - (\{\mu, 0)\}) \cap \mathscr{B}_{\zeta_n}$ such that $|\lambda_n - \mu| \leqslant \zeta_n < \xi$, $u_n \to 0$, but $|\langle \ell, u_n\rangle| \leqslant \eta \| u_n \|$. Dividing (0.1) by $\| u_n \|$ and letting $n \to \infty$, the form and properties of G imply that a subsequence of $u_n/\| u_n \|$ converges in E to v or $-v$. Hence $|\langle \ell, u_n\rangle|/\| u_n \| \to 1 > \eta$ along this subsequence, a contradiction. Thus there exists ζ_0 as above and in $\mathscr{B}_\zeta$ for $\zeta < \zeta_0$, if $(\lambda, u) \in \mathscr{S} - \{(\mu, 0)\}$, $u = \alpha v + w$ with $|\alpha| > \eta \| u \|$.

It is clear that $|\lambda - \mu| = 0(1)$ for α near 0. To show that $w = 0(|\alpha|)$, we argue similarly to the above. Note that $w = 0(|\alpha|)$ since $\| w \| \leqslant |\alpha| + \| u \| < |\alpha| + 1/\eta |\alpha|$. Let $(\lambda_n, u_n) \to (\mu, 0)$,

$$(\lambda_n, u_n) \in (\mathscr{S} - \{(\mu, 0)\}) \cap \mathscr{B}_\zeta \cap K^+_{\xi,\eta}.$$

Then $H(\lambda_n, \alpha_n v + w_n)/\| u_n \| \to 0$ as $n \to \infty$ and, as above, a subsequence of $u_n/\| u_n \|$ converges to v. Therefore, $|\langle \ell, u_n/\| u_n \| \rangle| = \alpha_n/\| u_n \| \to 1$ and $w_n/\| u_n \|$, $w_n/\alpha_n \to 0$ along this subsequence. Since this is true for all such subsequences, and likewise with $K^+_{\xi,\eta}$ replaced by $K^-_{\xi,\eta}$, it follows that $w = o(|\alpha|)$ for α near 0.

Remark. If we are dealing with a family of maps $\Phi(\lambda, u, t) = u - \lambda Lu - H(\lambda, u, t)$, $t \in [0, 1]$, and $H(\lambda, u, t)$ possesses the same properties as earlier uniformly in t, then ζ_0 can be chosen uniformly with respect to t.

Next using a simple reflection argument, we show near $(\mu, 0)$, $\mathscr{C}_\mu$ consists of two subcontinua which meet only at $(\mu, 0)$.

THEOREM 1.25. *$\mathscr{C}_\mu$ possesses a subcontinuum in $K^+_{\xi,\eta} \cup \{(\mu, 0)\}$ and in $K^-_{\xi,\eta} \cup \{(\mu, 0)\}$ each of which meet $(\mu, 0)$ and $\partial\mathscr{B}_\zeta$ for all $\zeta > 0$ sufficiently small.*

Proof. By Theorem 1.3 and Lemma 1.24, the result is true for at least one of the sets. Suppose it is not true for $K^-_{\xi,\eta} \cup \{(\mu, 0)\}$. Recall $u = \langle \ell, u \rangle v + w$. We define a new mapping $\hat{\Phi}(\lambda, u) \equiv u - \lambda Lu - \hat{H}(\lambda, u)$ as follows: for $-\eta \| u \| \geqslant \langle \ell, u \rangle$, let $\hat{H}(\lambda, u) = H(\lambda, u)$; for $0 \geqslant \langle \ell, u \rangle + \eta \| u \|$ and $\langle \ell, u \rangle \leqslant 0$, let

$$\hat{H}(\lambda, u) = \frac{-\langle \ell, u \rangle}{\eta \| u \|} H(\lambda, -\eta \| u \| v + w); \qquad \text{for} \quad \langle \ell, u \rangle \geqslant 0,$$

let $\hat{H}(\lambda, u) = -\hat{H}(\lambda, -u)$. Then $\hat{H}(\lambda, u)$ possesses the same properties as does $H(\lambda, u)$ and also is an odd function of u as is $\hat{\Phi}(\lambda, u)$. Consequently, for $\hat{\Phi}$ there exists a continuum $\hat{\mathscr{C}}_\mu$ satisfying the alternatives of Theorem 1.3. By Lemma 1.24, $\hat{\mathscr{C}}_\mu \cap \mathscr{B}_\zeta \subset K_{\xi,\eta} \cup \{(\mu, 0)\}$ for all $0 < \zeta < \zeta_1$. Therefore,

$$\hat{\mathscr{C}}_\mu \cap \partial\mathscr{B}_\zeta \cap K_{\xi,\eta} \neq \varnothing \qquad \text{for all} \quad 0 < \zeta < \zeta_1 . \tag{1.26}$$

On the other hand, since $\hat{\mathscr{C}}_\mu = \mathscr{C}_\mu$ in $K^-_{\xi,\eta}$ there exists values $\hat{\zeta} < \zeta_1$ such that $\hat{\mathscr{C}}_\mu \cap \partial\mathscr{B}_{\hat{\zeta}} \cap K^-_{\xi,\eta} = \varnothing$. The oddness with respect to u of $\hat{\Phi}$ then implies that $\hat{\mathscr{C}}_\mu \cap \partial\mathscr{B}_{\hat{\zeta}} \cap K^+_{\xi,\eta} = \varnothing$ contradicting (1.26). Thus it must be the case that $\mathscr{C}_\mu \cap \partial\mathscr{B}_\zeta \cap K^-_{\xi,\eta} \neq \varnothing$ for all $\zeta > 0$ small and the theorem is proved.

Let $\mathscr{D}_\mu^+(\mathscr{D}_\mu^-)$ denote the maximal subcontinuum of $\mathscr{C}_\mu$ which meets $(\mu, 0)$ and lies in $\overline{K^+_{\xi,\eta}(K^-_{\xi,\eta})}$. Let $\mathscr{C}_\mu^+(\mathscr{C}_\mu^-)$ be the maximal subcontinuum of $\overline{\mathscr{C}_\mu - \mathscr{D}_\mu^-(\mathscr{C}_\mu - \mathscr{D}_\mu^+)}$ which meets $(\mu, 0)$. Note that $\mathscr{C}_\mu^+ \supset \mathscr{D}_\mu^+$, $\mathscr{C}_\mu^- \supset \mathscr{D}_\mu^-$, and $\mathscr{C}_\mu = \mathscr{C}_\mu^+ \cup \mathscr{C}_\mu^-$. The subcontinua $\mathscr{C}_\mu^+$, $\mathscr{C}_\mu^-$ are extensions of those given by Theorem 1.25. A natural question to ask is: How global are these extensions? At least one of them must satisfy the alternatives of Theorem 1.3. If $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ meet outside of a neighborhood of $(\mu, 0)$, then $\mathscr{C}_\mu$ contains a "closed loop". This is the case, e.g., in the matrix example given earlier. Generalizing our reflection argument, we can give another partial answer to the above question.

THEOREM 1.27. *Each of $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ either satisfies the alternatives of Theorem* 1.3 *or* (iii) *contains a pair of points* (λ, u), $(\lambda, -u)$, $u \neq 0$.

Proof. Suppose $\mathscr{C}_\mu^-$ does not satisfy any of (i)–(iii). Then we can find a bounded open set $\mathscr{A} \subset \mathscr{E}$, containing $\mathscr{C}_\mu^- \cup \mathscr{B}_\zeta$, $\partial\mathscr{A} \cap \mathscr{S} = \varnothing$, and $\mathscr{A} - \mathscr{B}_\zeta$ contains no trivial solutions nor pairs of points of the form (λ, u), $(\lambda, -u)$, $\zeta < \zeta_0$ being as in Lemma 1.24. Let $\mathscr{O} = \mathscr{A} \cup \{(\lambda, u) \in \mathscr{E} \mid (\lambda, -u) \in \mathscr{A}\} \equiv \mathscr{A} \cup \tilde{\mathscr{A}}$. Note that $\mathscr{A} \cap \tilde{\mathscr{A}} = \mathscr{B}_\zeta$. A new mapping $\tilde{\Phi}(\lambda, u) = u - \lambda Lu - \tilde{H}(\lambda, u)$ is defined on $\bar{\mathscr{O}}$ by setting $\tilde{H}(\lambda, u) = \hat{H}(\lambda, u)$ in $\mathscr{B}_\zeta$, $\hat{H}$ as in Theorem 1.25, $\tilde{H}(\lambda, u) = H(\lambda, u)$ in $\mathscr{A} - \mathscr{B}_\zeta$ and $\tilde{H}(\lambda, u) = -H(\lambda, -u)$ in $\tilde{\mathscr{A}} - \mathscr{B}_\zeta$. Then $\tilde{H}$ possesses the same properties in $\bar{\mathscr{O}}$ as does H and is odd in u as in $\tilde{\Phi}$. Then $\tilde{\Phi}$ has no zeroes on $\partial\mathscr{O}$. But by Corollary 1.12 this is not possible. Thus $\mathscr{C}_\mu^-$ satisfies (i), (ii), or (iii).

Theorem 1.27 suffices for some of the applications given in Section 2. However, a stronger result is valid here, viz., each of $\mathscr{C}_\mu^+$, $\mathscr{C}_\mu^-$ satisfy the alternatives of Theorem 1.3. This does not imply that $\mathscr{C}_\mu$ consists of two globally distinct subcontinua since $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ may meet outside of a neighborhood of $(\mu, 0)$. If this does not happen, then we have globally distinct subcontinua. Unable to give a proof of this by using the previous theory, we proceed to present an independent proof of this result. This, in fact, amounts to an alternate proof of Theorem 1.3 for the simple characteristic value case and is related to the local theorem when H is Fréchet differentiable mentioned after Theorem 1.16.

Recalling that $E = \mathbf{R} \oplus E_1$, (0.1) can be rewritten in an equivalent form as an equation in $\mathbf{R} \oplus E_1$. Thus,

$$\alpha = \frac{\alpha\lambda}{\mu} + \langle \ell, H(\lambda, \alpha v + w)\rangle \tag{1.28}$$

and

$$w = \lambda Lw + (I - P)\, H(\lambda, \alpha v + w), \tag{1.29}$$

with $u = \alpha v + w$, $\alpha = \langle \ell, u\rangle$, $w \in E_1$, and $Pu = \alpha v$. Since $(I - \mu L)^{-1}$ exists as a bounded operator on E_1, Eq. (1.29) is equivalent to

$$w = (\lambda - \mu)(I - \mu L)^{-1} Lw + (I - \mu L)^{-1} (I - P)\, H(\lambda, \alpha v + w) \equiv T(\alpha, \lambda, w). \tag{1.30}$$

Let $\hat{\mathscr{E}} = \mathbf{R} \times \mathbf{R} \times E_1$ and $\hat{E} = \mathbf{R} \times E_1$. We define

$$\Psi(\alpha, \lambda, w) = (\lambda - t(\alpha, \lambda, w), w - T(\alpha, \lambda, w)),$$

where

$$t(\alpha, \lambda, w) = \alpha\lambda/\mu + \langle \ell, H(\lambda, \alpha v + w)\rangle - \alpha + \lambda.$$

Then $\Psi : \hat{\mathscr{E}} \to \hat{E}$. Since $(t, T) : \hat{\mathscr{E}} \to \hat{E}$ is continuous and compact, for fixed α, Ψ is of the appropriate form for the use of the theory of Leray–Schauder degree. Note that any zero of Ψ is a solution of (0.1) and conversely. The trivial solutions of (0.1) correspond to the solutions $(0, \lambda, 0)$, $\lambda \in \mathbf{R}$, of Eqs. (1.28) and (1.30). In what follows, when the (λ, w) dependence of Ψ is not important, we just write $\Psi(\alpha)$.

Let $\hat{\mathscr{S}}$ denote the closure of the set of nontrivial solutions of Eqs. (1.28) and (1.30).

First we find subcontinua of $\hat{\mathscr{S}}$ near $(0, \mu, 0)$. Let

$$\mathscr{U} = \{(\lambda, w) \in \hat{E} \mid |\lambda - \mu| < \epsilon_1, \|w\| < \rho_1\}.$$

By Lemma 1.24 it can be assumed that all nontrivial solutions (α, λ, w) of Eqs. (1.28) and (1.30) near $(0, \mu, 0)$ have $(\lambda, w) \in \mathscr{U}$ if $|\alpha| \leqslant \alpha_1$ provided that $\alpha_1, \epsilon_1, \rho_1$ are sufficiently small. Therefore $\Psi(\alpha, \lambda, w) \neq 0$ on $\partial\mathscr{U}$ for $0 < |\alpha| \leqslant \alpha_1$. (The trivial solutions pierce $\partial\mathscr{U}$ at $\alpha = 0$). Consequently, $d(\Psi(\alpha), \mathscr{U}, (0, 0)) \equiv d(\Psi(\alpha), \mathscr{U})$ is well-defined for $0 < |\alpha| \leqslant \alpha_1$. Since for any $\alpha_2 \in (0, \alpha_1)$, $[\alpha_2, \alpha_1] \times \mathscr{U}$ is a bounded open set in $[\alpha_2, \alpha_1] \times \hat{E}$, as in Theorem 1.3, the homotopy invariance of d implies that

$$d(\Psi(\alpha), \mathscr{U}) \equiv \text{constant} = c_\nu, \tag{1.31}$$

for $0 < \nu\alpha \leqslant \alpha_1$, $\nu = +, -$.

To evaluate c_ν, consider the family of operators $\Psi_\theta(\alpha, \lambda, w) = (\lambda - (\alpha/\mu + 1)\lambda - \theta\langle \ell, H(\lambda, \alpha v + w)\rangle + \alpha, w - \theta T(\alpha, \lambda, w))$. It can be assumed that $\Psi_\theta(\alpha) \neq 0$ on $\partial\mathscr{U}$ for $0 \leqslant \theta \leqslant 1$, $0 < |\alpha| \leqslant \alpha_1$. Using the homotopy invariance of d again gives

$$d(\Psi_\theta(\alpha), \mathscr{U}) = c_\nu \qquad 0 \leqslant \theta \leqslant 1, \tag{1.32}$$

with $0 < \nu\alpha \leqslant \alpha_1$, $\nu = +, -$. Hence to calculate c_ν, it suffices to take $\theta = 0$, when $\Psi_0(\alpha, \lambda, w) = (\lambda - (\alpha/\mu + 1)\lambda + \alpha, w)$ is linear (and inhomogeneous). The only zero $\Psi_0(\alpha)$ possesses is $\lambda = \mu$, $w = 0$. Moreover, $\Psi_0(\alpha)$ is an isomorphism on $\hat{E}$. Hence

$$c_\nu = i(\Psi_0(\alpha), (\mu, 0)) = 1 \quad \text{or} \quad -1.$$

For what follows it is important to know that $c_+ \neq c_-$ which will be shown next. Observe that

$$i(\Psi_0(\alpha), (\mu, 0)) = d(\Psi_0(\alpha), \mathscr{U}, (0, 0)) = d(\hat{\Psi}_0(\alpha), \mathscr{U}, (-\alpha, 0)), \quad (1.33)$$

where

$$\hat{\Psi}_0(\alpha, \lambda, w) = (\lambda - (\alpha/\mu + 1)\,\lambda, w), \qquad \text{i.e.,} \quad \hat{\Psi}_0(\alpha)$$

is the homogeneous linear operator corresponding to $\Psi_0(\alpha)$. Since $\hat{\Psi}_0(\alpha)$ is an isomorphism ($\alpha \neq 0$),

$$d(\hat{\Psi}_0(\alpha), \mathscr{U}, (-\alpha, 0) = i(\hat{\Psi}_0(\alpha), (-\alpha, 0)) = i(\hat{\Psi}_0(\alpha), (0, 0)). \quad (1.34)$$

Since $(\lambda - \gamma(\alpha/\mu + 1)\lambda, w)$ possesses no characteristic values γ in the interval $(0, 1)$ for $\alpha < 0$ and one characteristic value in this interval for $\alpha > 0$, the basic theorem on change of index, [1–3, Appendix] and Eqs. (1.32) and (1.34) imply that $c_- = 1$, $c_+ = -1$.

By a slight modification of the arguments of Theorem 1.3, it follows that $\mathscr{S}$ contains a pair of continua $\mathscr{M}_\mu^\nu$, $\nu = +, -$, each of which meets $(0, \mu, 0)$ and $\{\nu\alpha_1\} \times \mathscr{U}$, and lies in $\{(\alpha, \mathscr{U}) \subset \hat{\mathscr{E}} \mid 0 \leqslant \nu\alpha \leqslant \alpha_1\}$. These continua can essentially be identified with $\overline{\mathscr{D}_\mu^\nu \text{ as defined}}$ earlier. Let $\mathscr{N}_\mu^\nu$ be the maximal subcontinuum of $\hat{\mathscr{S}} - \mathscr{M}_\mu^{-\nu}$ which meets $(0, \mu, 0)$ and let $\mathscr{N}_\mu = \mathscr{N}_\mu^+ \cup \mathscr{N}_\mu^-$. Then $\mathscr{N}_\mu$, $\mathscr{N}_\mu^\nu$ can be identified with $\mathscr{C}_\mu$, $\mathscr{C}_\mu^\nu$ via the isomorphism $(\alpha, \lambda, w) \leftrightarrow (\lambda, u) = (\lambda, \alpha v + w)$.

Next we show each of $\mathscr{N}_\mu^+$, $\mathscr{N}_\mu^-$ meets ∞ in $\hat{\mathscr{E}}$ or meets $(0, \tilde{\mu}, 0)$, where $\mu \neq \tilde{\mu} \in r(L)$. Suppose, e.g., $\mathscr{N}_\mu^+$ does not satisfy either of these alternatives. There are two cases to consider. First assume that $\mathscr{N}_\mu^+$ meets $\mathscr{N}_\mu^-$ outside of a neighborhood of $(0, \mu, 0)$. As in Lemma 1.2, we can find a bounded open set $\mathscr{O} \supset \mathscr{N}_\mu$ such that $\partial\mathscr{O} \cap \hat{\mathscr{S}} = \varnothing$ and $\mathscr{O}$ consists of $[-\alpha_1, \alpha_1] \times \mathscr{U}$ near $(0, \mu, 0)$. In addition, it can be assumed that $\mathscr{O}$ contains no trivial solutions other than those in $\{0\} \times \mathscr{U}$. Let $\mathscr{O}_\alpha = \{(\lambda, w) \in \hat{E} \mid (\alpha, \lambda, w) \in \mathscr{O}\}$. Then $\mathscr{O}_\alpha = \varnothing$ for $|\alpha|$ sufficiently large and by the homotopy invariance of d,

$$d(\Psi(\alpha), \mathscr{O}_\alpha) = 0 \qquad \alpha \neq 0. \quad (1.35)$$

For $0 < \alpha \leqslant \alpha_1$, the additivity of d implies

$$d(\Psi(\alpha), \mathscr{O}_\alpha) = d(\Psi(\alpha), \mathscr{U}) + d(\Psi(\alpha), \mathscr{O}_\alpha - \overline{\mathscr{U}}). \quad (1.36)$$

Hence Eqs. (1.32), (1.35), and (1.36) yield

$$d(\Psi(\alpha), \mathscr{O}_\alpha - \overline{\mathscr{U}}) = -c_+ = 1 \qquad 0 < \alpha \leqslant \alpha_1. \quad (1.37)$$

Since $\Psi(\alpha) \neq 0$ on $\partial(\mathcal{O} - [-\alpha_1, \alpha_1] \times \mathcal{U})$ (∂ in $[-\alpha_1, \alpha_1] \times \hat{E}$)

$$d(\Psi(\alpha), \mathcal{O}_\alpha - \bar{\mathcal{U}}) = 1 \qquad |\alpha| \leqslant \alpha_1 . \tag{1.38}$$

But then (1.36) implies

$$d(\Psi(\alpha), \mathcal{O}_\alpha) = 2 \neq 0 \tag{1.39}$$

for $0 > \alpha \geqslant -\alpha_1$, contradicting Eq. (1.35).

Next assume that $\mathcal{N}_\mu^+$ does not meet $\mathcal{N}_\mu^-$ outside of a neighborhood of $(0, \mu, 0)$. Then $\mathcal{O}$ as above can be chosen with the modifications that $\mathcal{A} \equiv \{-\alpha_1/2\} \times \mathcal{U} \subset \partial\mathcal{O}$, $[-\alpha_1, -\alpha_1/2] \times \mathcal{U} \cap \mathcal{O} = \varnothing$, and $(\partial\mathcal{O} - \mathcal{A}) \cap \hat{\mathcal{S}} = \varnothing$. Note that $\mathcal{A} \cap \hat{\mathcal{S}} \subset \mathcal{N}_\mu^-$. Equation (1.35) is now valid for $\alpha > 0$ and $\alpha < -\alpha_1/2$. The computations (1.36)–(1.38) are unchanged. Since $\mathcal{O}_\alpha - \mathcal{U} = \mathcal{O}_\alpha$ for $\alpha = -3\alpha_1/4$, Eqs. (1.35) and (1.38) are inconsistent. Thus $\mathcal{N}_\mu^+$ satisfies the above alternatives.

Identifying $\hat{\mathscr{E}}$ with $\mathscr{E}$, we have shown

THEOREM 1.40. *Each of $\mathscr{C}_\mu^+$, $\mathscr{C}_\mu^-$ meets $(\mu, 0)$ and either*

(i) *meets ∞ in $\mathscr{E}$, or*

(ii) *meets $(\tilde{\mu}, 0)$, where $\mu \neq \tilde{\mu} \in r(L)$.*

Remark. We suspect that there is an analog of this theorem for the general odd multiplicity case.

2. SOME APPLICATIONS

Here we will give some applications of the theory of Section 1 to ordinary differential equations and integral equations as well as to quasilinear elliptic partial differential equations.

Our first application to nonlinear Sturm–Liouville problems for second-order ordinary differential equations has already been done in [3] but the proof given here is simpler. Consider

$$\mathscr{L}u \equiv -(pu')' + qu = F(x, u, u', \lambda) \qquad 0 < x < \pi \tag{2.1}$$

and

$$a_0 u(0) + b_0 u'(0) = 0, \qquad a_1 u(\pi) + b_1 u'(\pi) = 0,$$

where $(a_0^2 + b_0^2)(a_1^2 + b_1^2) \neq 0$. The boundary conditions of Eq. (2.1) will henceforth be denoted by B.C. The function $F(x, \xi, \eta, \lambda) = \lambda a(x)\xi + H(x, \xi, \eta, \lambda)$, and H is $0((\xi^2 + \eta^2)^{1/2})$ near $(\xi, \eta) = (0, 0)$ uniformly on bounded λ intervals. In addition, p, q, a are assumed respectively to be continuously differentiable and positive, continuous, continuous and positive on $[0, \pi]$, and F is assumed to be continuous in its arguments on $[0, \pi] \times \mathbf{R}^3$.

If $H \equiv 0$, Eq. (2.1) becomes a linear Sturm–Liouville problem:

$$\mathscr{L}u = \lambda au, \qquad 0 < x < \pi; \qquad u \in \text{B.C.} \tag{2.2}$$

As is well-known, Eq. (2.2) possesses an increasing sequence of simple eigenvalues $\lambda_1 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. Any eigenfunction v_n corresponding to λ_n has exactly $n - 1$ simple nodal zeroes on $(0, \pi)$. (By a nodal zero we mean the function changes sign at the zero and at a simple nodal zero, the derivative of the function is nonzero).

To exploit these nodal properties an appropriate family of sets is introduced. Let E denote the Banach space $C^1[0, \pi] \cap$ B.C. with the usual norm $\| u \|_1 = \max_{x \in [0,\pi]} | u(x)| + \max_{x \in [0,\pi]} | u'(x)|$. Let S_k^+ be the set of $u \in E$ which have exactly $k - 1$ simple nodal zeroes on $(0, \pi)$ and which are positive for $0 \neq x$ near 0; then, $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. The sets S_k^+, S_k^+, S_k are open in E. The eigenfunction v_k corresponding to λ_k in Eq. (2.2) is made unique by requiring that $v_k \in S_k^+$ and $\| v_k \|_1 = 1$.

Let $\mathscr{E} = \mathbf{R} \times E$, $\mathscr{S}_k^+ = \mathbf{R} \times S_k^+$, $\mathscr{S}_k^- = \mathbf{R} \times S_k^-$, and $\mathscr{S}_k = \mathbf{R} \times S_k$. The linear existence theory for Eq. (2.2) can be stated as: For each integer $k > 0$ and each $\nu = +$ or $-$, there exists a half line of solutions of Eq. (2.2) in $\mathscr{S}_k^\nu$ of the form $(\lambda_k, \alpha v_k)$, $\alpha \in \mathbf{R}^\nu$. This half line joins $(\lambda_k, 0)$ to infinity in $\mathscr{E}$. (Here $\mathbf{R}^\nu = (\lambda \in \mathbf{R} \mid 0 \leqslant \nu\lambda \leqslant \infty\}$, $\nu = +, -$).

An analogous result holds for Eq. (2.1).

THEOREM 2.3. *For each integer $k > 0$ and each $\nu = +$ or $-$, there exists a continuum of solutions of Eq.* (2.1) *in $\mathscr{S}_k^\nu \cup \{(\lambda_k, 0)\}$ which meets $(\lambda_k, 0)$ and ∞ in $\mathscr{E}$.*

Proof. Note first that if (λ, u) is a solution of Eq. (2.1) and u has a double zero, then the growth estimate on H near the double zero and linearity of $\mathscr{L}$ and au implies that $u \equiv 0$ on $[0, \pi]$. Therefore, in particular, any solution (λ, u) of Eq. (2.1) with $u \in \partial S_k^\nu$ has $u \equiv 0$.

Assume that 0 is not an eigenvalue of $\mathscr{L}$. Then using the Green's function $g(x, y)$ of $\mathscr{L}$ with respect to the B.C. of Eq. (2.1), the equation can be converted to the equivalent integral equation

$$u(x) = \int_0^\pi g(x, y) F(y, u(y), u'(y), \lambda)\, dy \equiv G(\lambda, u). \tag{2.4}$$

It is easily seen that G is a continuous compact map of $\mathscr{E} \to E$. Hence Eq. (2.4) is of the form (0. 1) with $Lu = \int_0^\pi g(x, y)\, a(y)\, u(y)\, dy$. The eigenvalues of $\mathscr{L}$ are the characteristic values of L and are simple. Therefore the hypotheses of Theorem 1.3 are satisfied and there exists a continuum $\mathscr{C}_{\lambda_k} \equiv \mathscr{C}_k$ as in Theorem 1.3. Lemma 1.24 implies that if $(\lambda, u) \in \mathscr{C}_k$ and is near $(\lambda_k, 0)$, $u = \alpha v_k + w$ with $w = 0(|\alpha|)$. Since S_k^ν is open and $v_k \in S_k$, then

$$(\lambda, u) \in \mathscr{S}_k \qquad \text{and} \qquad (\mathscr{C}_k - \{(\lambda_k, 0)\}) \cap \mathscr{B}_\zeta \subset \mathscr{S}_k$$

for all $0 < \zeta$ small. By an above remark, $(\mathscr{C}_k - \{(\lambda_k, 0)\}) \cap \partial\mathscr{S}_k = \varnothing$. Consequently, $\mathscr{C}_k$ lies in $\mathscr{S}_k \cup \{(\lambda_k, 0)\}$ and alternative (ii) of Theorem 1.3 is not possible.

It remains to decompose $\mathscr{C}_k$ into two subcontinua which meet $(\lambda_k, 0)$ and ∞ in $\mathscr{S}_k^+ \cup \{(\lambda_k, 0)\}$ and $\mathscr{S}_k^- \cup \{(\lambda_k, 0)\}$, respectively. Again writing $u = \alpha v_k + w$ for $(\lambda, u) \in \mathscr{C}_k - \{(\lambda_k, 0)\}$ and near $(\lambda_k, 0)$, we have $\alpha v_k \in \mathscr{S}_k^\nu$ if $0 \neq \alpha \in \mathbf{R}^\nu$ and, therefore,

$$(\mathscr{C}_k^+ - \{(\lambda_k, 0)\}) \cap \mathscr{B}_\zeta \subset \mathscr{S}_k^+,$$

$\mathscr{C}_k^- - \{(\lambda_k, 0)\} \cap \mathscr{B}_\zeta \subset \mathscr{S}_k^-$ for all $0 < \zeta$ small. Since $\mathscr{C}_k^\nu - \{(\lambda_k, 0)\}$ cannot leave $\mathscr{S}_k^\nu$ outside of a neighborhood of $(\lambda_k, 0)$ and $\mathscr{S}_k^\nu$ does not contain a pair of points of the form (λ, u), $(\lambda, -u)$, it follows from Theorem 1.27 or Theorem 1.40 that $\mathscr{C}_k^\nu$ meets infinity in $\mathscr{S}_k^\nu$, $\nu = +, -$.

If 0 is an eigenvalue of $\mathscr{L}$, the result is trivial if $\lambda_k = 0$. If $\lambda_k \neq 0$, then replacing $\mathscr{L}$ by $\mathscr{L} + \epsilon a$, and passing to a limit using the already established result and the compactness of G, completes the proof of Theorem 2.3 (see [3]).

Remark. It is possible to generalize Theorem 2.3 by permitting $\mathscr{L}$ to depend on λ and on u in a nonlinear fashion (see [3]). Likewise F could be a map of $C^1[0, \pi] \times \mathbf{R} \to C[0, \pi]$. However, then, the form of F must be such that Eq. (2.1) has the property that whenever (λ, u) is a solution of (2.1) with u having a double zero, then $u \equiv 0$ (see [10]).

Next we show how nodal properties can likewise be exploited for a class of nonlinear integral equations. Consider

$$u(x) = \lambda \int_0^1 K(x, y) F(y, u(y))\, u(y)\, dy \equiv G(\lambda, u), \tag{2.5}$$

where $K(x, y)$ is a continuous symmetric oscillation kernel on $[0, 1]^2$ and $F(y, z)$ is a positive continuous function on $[0, 1] \times \mathbf{R}$. Let

$E = C[0, 1]$ under $\| u \| = \max_{x \in [0,1]} | u(x)|$. Then G is continuous and compact on $\mathscr{E} = \mathbf{R} \times E$ as is

$$Lu = \int_0^\pi K(x, y) F(y, 0) u(y) \, dy$$

and $H(\lambda, u) = G(\lambda, u) - \lambda Lu$ is $0(\| u \|)$ for $\| u \|$ near 0.

Since $K(x, y) F(z, 0)$ is also a symmetrizable oscillation kernel, the linear characteristic value problem

$$u = \lambda Lu \tag{2.6}$$

possesses an increasing sequence of positive simple characteristic values $\lambda_1 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$ (see [9]). Any eigenfunction v_k corresponding to λ_k has exactly $k - 1$ nodal zeroes on (0, 1). Let N_k^+ denote the subset of $u \in E$ such that u has exactly $k - 1$ nodal zeroes on (0, 1) and is positive near $x = 0$; $N_k^- = -N_k^+$, and $N_k = N_k^+ \cup N_k^-$. Then $v_k \in N_k$. We normalize v_k by requiring $\| v_k \| = 1$ and $v_k \in N_k^+$.

As was the case with Eqs. (2.1) and (2.2), there is a nonlinear analog for Eq. (2.5) of the linear theory for Eq. (2.6). Let $\mathscr{N}_j^\nu = \mathbf{R} \times N_j^\nu$ for $\nu = +, -$, and $\mathscr{N}_j = \mathbf{R} \times N_j$.

THEOREM 2.7. *For each $k > 0$ and each $\nu = +$ or $-$, there exists a continuum of solutions of Eq. (2.5) in $\mathscr{N}_k^\nu$ which meets $(\lambda_k, 0)$ and ∞ in $\mathscr{E}$.*

Proof. By Theorem 1.3, there exists a continuum $\mathscr{C}\lambda_k \equiv \mathscr{C}_k \subset \mathscr{S}$ meeting $(\lambda_k, 0)$ and satisfying the alternatives of Theorem 1.3. Let $\mathfrak{A} = \{(\lambda, u) \in \mathscr{C}_k \mid (\lambda, u) \in \mathscr{N}_k \cup \{(\lambda_k, 0)\}\}$. We will show that $\mathfrak{A}$ is both open and closed in $\mathscr{C}_k$ under the induced topology from $\mathscr{E}$ and therefore since $\mathscr{C}_k$ is a continuum, $\mathfrak{A} \equiv \mathscr{C}_k$. (Certainly $\mathfrak{A} \neq \varnothing$).

Suppose $(\lambda, u) \in \mathfrak{A}$, $(\lambda, u) \neq (\lambda_k, 0)$. Since $K(x, y) F(y, u(y))$ is a symmetrizable oscillation kernel, and (λ, u) is an eigenpair of Eq. (2.5) with $u \in N_k$, it follows that λ is the k-th characteristic value of $K(x, y) F(y, u(y))$. Likewise, this is the case if $(\lambda, u) = (\lambda_k, 0)$.

Now we show $\mathfrak{A}$ is closed, for if $(\mu_n, u_n) \subset \mathfrak{A}$ and $(\mu_n, u_n) \to (\mu, u)$, then $(\mu, u) \in \mathscr{C}_k$ since it is a closed set. The kernels $KF(y, u_n) \to KF(y, u)$ in E and therefore the corresponding integral operators converge in the operator norm. Consequently the respective sets of characteristic value of these operators converge uniformly on compact subsets of $\mathbf{R}$. Hence μ being the limit of k-th characteristic values must be the k-th characteristic value of $KF(y, u)$. Therefore, u is a k-th eigenvector and $u \in N_k$ unless $u = 0$ in which case $\lambda = \lambda_k$. Thus $\mathfrak{A}$ is closed.

A similar argument shows $\mathcal{O}$ is open. Let $(\lambda, u) \in \mathcal{O}$. Since λ is the k-th characteristic value of $KF(y, u)$, given any $\epsilon > 0$, there exists a $\rho > 0$ such that if $\| u - w \| < \rho$ the k-th characteristic value μ of $KF(y, w)$ satisfies $| \lambda - \mu | < \epsilon$. Moreover, since the kernels $KF(y, \varphi)$ possess only simple eigenvalues, given any $\epsilon_1 > 0$ there exists $\rho_1 > 0$ such that $\| u - w \| < \rho_1$ implies all eigenvalues γ of $KF(y, w)$ other than the k-th satisfy $| \gamma - \lambda | \geqslant \epsilon_1$. Taking $\epsilon_1 = 2\epsilon$, we see that if $(\mu, w) \in \mathscr{C}_k$ and $| \mu - \lambda | < \epsilon$, $\| w - u \| < \min(\rho, \rho_1)$, then $(\mu, w) \in \mathscr{N}_k \cup \{(\lambda_k, 0)\}$. Hence $\mathcal{O}$ is open.

By our foregoing remark it follows that $\mathcal{O} \equiv \mathscr{C}_k$. Therefore, if $k \neq j$, $\mathscr{C}_k \cap \mathscr{C}_j = \varnothing$ and alternative (ii) of Theorem 1.3 is not possible. Thus alternative (i) prevails.

Next $\mathscr{C}_k$ is decomposed into two subcontinua lying in

$$\mathscr{N}_k^+ \cup \{(\lambda_k, 0)\}, \mathscr{N}_k^- \cup \{(\lambda_k, 0)\},$$

respectively, each meeting ∞. By the remarks following Theorem 1.16, we can break up $\mathscr{C}_k$ into $\mathscr{C}_k^+$, $\mathscr{C}_k^-$. It will be shown that

$$\mathscr{C}_k^\nu \subset \mathscr{N}_k^\nu \cup \{(\lambda_k, 0)\}, \qquad \nu = +, -.$$

Hence $\mathscr{C}_k^+$ does not meet $\mathscr{C}_k^-$ outside of a neighborhood of $(\lambda_k, 0)$ and neither of $\mathscr{C}_k^+, \mathscr{C}_k^-$ contains a pair of points (λ, u), $(\lambda, -u)$. Therefore, by Theorem 1.27 or Theorem 1.40, $\mathscr{C}_k^\nu$ meets ∞ in $\mathscr{N}_k^\nu$, $\nu = +, -$.

Thus to complete the proof, we show $\mathscr{C}_k^\nu \subset \mathscr{N}_k^\nu \cup \{(\lambda_k, 0)\}$, $\nu = +, -$. First observe that $\mathscr{C}_k^+$ cannot have nonempty intersection with both $\mathscr{N}_k^+$ and $\mathscr{N}_k^-$ for otherwise since $\mathscr{C}_k^+$ is a continuum, we could find a point (λ, u) on $\mathscr{C}_k^+$ with $u \in N_j$, $j < k$, which is not possible. Therefore $\mathscr{C}_k^+$ lies either in $\mathscr{N}_k^+ \cup \{(\lambda_k, 0)\}$ or in $\mathscr{N}_u^- \cup \{(\lambda_k, 0)\}$. Suppose $\mathscr{C}_k^+ \subset \mathscr{N}_k^- \cup \{(\lambda_k, 0)\}$. Let $(\lambda, u) \in \mathscr{C}_k^+$, (λ, u) near $(\lambda_k, 0)$. Then $u = \alpha v_k + w$, where $0 < \alpha = \langle \ell, u \rangle$ and $w \in E_1$. Since $u \in N_k^-$, $u/\alpha = v_k + w/\alpha \in N_k^+$. Letting $\alpha \to 0$ and using Lemma 1.24, we find $u/\alpha \to v_k \in \bar{N}_k^-$. But $v_k \in N_k^+$ and

$$N_k^+ \cap \bar{N}_k^- = \varnothing.$$

Thus $\mathscr{C}_k^+ \subset \mathscr{N}_k^+ \cup \{(\lambda_k, 0)\}$. Similarly $\mathscr{C}_k^- \subset \mathscr{N}_k^- \cup \{(\lambda_k, 0)\}$, and the proof is complete.

Remark 1. The only point in the above proof in which the symmetry of K has played a role is in guaranteeing that certain linear operators have positive simple eigenvalues and corresponding eigenfunctions with nodal properties. Thus Theorem 2.7 is also valid if K

is an arbitrary oscillation kernel provided its eigenvalues and eigenfunctions have these properties as, e.g., in [19].

Remark. 2. The ideas used in the above proof can readily be generalized to include the case in which F is a map depending also on λ [10]. Likewise it is possible to generalize the results of Parter on interlocking pairs of ordinary differential equations of a special form [20].

The last application in this section treats a nonlinear eigenvalue problem for a class of quasilinear elliptic partial differential equations. Let $\mathscr{D}$ be a smooth bounded domain in $\mathbf{R}^n$. Consider the boundary value problem:

$$\begin{aligned} \mathscr{L}u \equiv & -\sum_{i,j=1}^{n} a_{ij}(x, u, Du)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x, u, Du)\, u_{x_i} \\ & + c(x, u, Du)\, u = \lambda(a(x)\, u + F(x, u, Du, \lambda)) \qquad x \in \mathscr{D} \qquad (2.8) \\ & u = 0 \qquad \text{on } \partial\mathscr{D}. \end{aligned}$$

Here Du denotes arbitrary first partial derivatives of u. The functions a_{ij}, b_i, c, a, F are assumed to be continuously differentiable functions of their arguments. In addition, we assume $c \geqslant 0$, $a \geqslant a_0 > 0$, $F \geqslant 0$, $F(x, u, p, \lambda) = 0((u^2 + |p|^2)^{1/2})$ near $(u, p) = (0, 0)$ $(p \in \mathbf{R}^n)$ uniformly on bounded λ intervals, and that Eq. (2.8) is uniformly elliptic, i.e.,

$$\sum a_{ij}(x, \eta, p)\, \xi_i \xi_j \geqslant \beta\, |\xi|^2$$

for all $x \in \mathscr{D}$, $\eta \in \mathbf{R}$, p, $\xi \in \mathbf{R}^n$ with β a positive constant.

Let $\alpha \in (0, 1)$ and let E be the Banach space: $E = \{u \in \mathscr{C}^{1+\alpha}(\bar{\mathscr{D}}) \mid u = 0 \text{ on } \partial\mathscr{D}\}$ under the norm

$$\begin{aligned} \| u \|_{1+\alpha} = & \max_{x \in \mathscr{D}} |u(x)| + \max_{1 \leqslant i \leqslant n} \max_{x \in \mathscr{D}} |u_{x_i}(x)| \\ & + \max_{1 \leqslant i \leqslant n} \max_{x, y \in \mathscr{D}} |u_{x_i}(x) - u_{x_i}(y)| / |x - y|^{\alpha}. \end{aligned}$$

Let $P^+ = \{u \in E \mid u > 0 \text{ in } \mathscr{D} \text{ and } \partial u/\partial\omega < 0 \text{ on } \partial\mathscr{D}\}$ and $P^- = -P^+$, where ω is the outward pointing normal to $\partial\mathscr{D}$. P^+ and P^- are open subsets of E.

Let $\mathscr{E} = \mathbf{R} \times E$, $\mathscr{P}^\nu = \mathbf{R} \times P^\nu$, $\nu = +, -$. We will prove the existence of a continuum of solutions of Eq. (2.8) with $u \in P^+$. We define a mapping of $\mathscr{E} \to E$ as follows: For $(\lambda, u) \in \mathscr{E}$, let $v =$

$G(\lambda, u)$ be the solution of the linear uniformly elliptic partial differential equation:

$$
\begin{aligned}
-\sum_{i,j=1}^{n} a_{ij}(x, u, Du)\, v_{x_i x_j} + \sum_{i=1}^{n} b_i(x, u, Du)\, v_{x_i} + c(x, u, Du)\, v \\
= \lambda(a(x)\, u + F(x, u, Du, \lambda)) \qquad x \in \mathscr{D} \qquad (2.9) \\
v = 0 \qquad \text{on} \quad \partial\mathscr{D}.
\end{aligned}
$$

The standard linear existence theory for such equations implies there exists a unique $v \in \mathscr{C}^{2+\alpha}(\bar{\mathscr{D}})$ satisfying Eq. (2.9) (see [11]). The Schauder estimates imply that the mapping G is continuous and compact. Any solution (λ, u) of Eq. (2.8) satisfies $u = G(\lambda, u)$, and conversely.

For $(\lambda, u) \in \mathscr{E}$, let $w \equiv T(\lambda, u)$ denote the solution of

$$
\begin{aligned}
-\sum_{i,j=1}^{n} a_{ij}(x, 0, 0)\, w_{x_i x_j} + \sum_{i=1}^{n} b_i(x, 0, 0)\, w_{x_i} + c(x, 0, 0)\, w = \lambda a u \qquad x \in \mathscr{D} \\
w = 0 \qquad \text{on} \quad \partial\mathscr{D}.
\end{aligned}
\tag{2.10}
$$

Then $T(\lambda, u) = \lambda L u$, where, as above, L is the linear compact map of $E \to E$. It is easily seen that $H(\lambda, u) \equiv G(\lambda, u) - \lambda L u$ is $O(\| u \|_{1+\alpha})$ for u near O uniformly on bounded λ intervals.

Consider the linear characteristic value problem $v = \lambda L v$, i.e.,

$$
\begin{aligned}
-\sum_{i,j=1}^{n} a_{ij}(x, 0, 0)\, v_{x_i x_j} + \sum_{i=1}^{n} b_i(x, 0, 0)\, v_{x_i} + c(x, 0, 0)\, v = \lambda a(x)\, v \qquad x \in \mathscr{D} \\
v = 0 \qquad \text{on} \quad \partial\mathscr{D}.
\end{aligned}
\tag{2.11}
$$

Using the maximum principle and the linear theory for L, it is easily seen that L is a strongly positive operator (in the sense of Krein–Rutman [11]) on the cone $\bar{P}^+$. Therefore by a theorem of Krein–Rutman, the smallest characteristic value λ_1 of L is positive and simple and possesses a corresponding eigenfunction $v_1 \in P^+$. v_1 is made unique by taking $\| v_1 \|_{1+\alpha} = 1$.

Thus we see that all of the hypotheses of Theorem 1.3 are satisfied for Eq. (2.8) with $\mu = \lambda_1$ and by that theorem, there exists a continuum $\mathscr{C}_{\lambda_1} \equiv \mathscr{C}_1$ of solutions (λ, u) of Eq. (2.8) in $\mathscr{E}$ satisfying the alternatives of that theorem. If $(\lambda, u) \in \mathscr{C}_1$ and is near $(\lambda_1, 0)$, then $u = \gamma v_1 + w$

where $|\gamma|$ is small and $w = o(|\gamma|)$. Hence as for Eq. (2.1), $u \in P^\nu$ if $\gamma \in \mathbf{R}^\nu$, $\nu = +, -$, and $|\gamma|$ small. Thus $\mathscr{C}_1{}^\nu \cap \mathscr{B}_\zeta \subset \mathscr{P}^\nu \cup \{(\lambda_1, 0)\}$ for $0 < \zeta$ small, $\nu = +, -$.

Next we claim $\mathscr{C}_1{}^+ \subset \mathscr{P}^+$ and therefore as in Theorem 2.3 or 2.7, $\mathscr{C}_1{}^+$ meets ∞ in $\mathscr{E}$. To prove this, note first we can assume $(\lambda, u) \in \mathscr{C}_1$ implies $\lambda > 0$ for otherwise $\lambda = 0$ is an eigenvalue of an equation of the form (2.8). But this is impossible via the maximum principle. If $\mathscr{C}_1{}^+ \not\subset \mathscr{P}^+$, there exists $(\lambda, u) \in \mathscr{C}_1{}^+ \cap (\mathbf{R}^+ \times \partial P^+)$, $(\lambda, u) \neq (\lambda_1, 0)$, such that (λ, u) is the limit in $\mathscr{E}$ of $(\lambda_n, u_n) \in \mathscr{P}^+$. The function u has either an interior zero in $\mathscr{D}$ or $\partial u/\partial\omega = 0$ at some point on $\partial\mathscr{D}$, $\partial/\partial\omega$ denoting the outward normal derivative to $\partial\mathscr{D}$. Then from Eq. (2.8), $\mathscr{L}u \geqslant 0$ and the strong maximum principle [13] implies $u \equiv 0$. Therefore $(\lambda, 0) \in \mathscr{C}_1{}^+$ and is the limit of solutions of Eq. (2.8) in $\mathscr{P}^+$. An argument similar to that of Lemma 1.24 then shows that $\lambda = \lambda_1$ contrary to hypothesis. Thus we have proved

THEOREM 2.12. *There is a continuum of solutions $\mathscr{C}_1{}^+$ of Eq.* (2.8) *in $\mathscr{P}^+ \cup \{(\lambda_1, 0)\}$ which meets $(\lambda_1, 0)$ and ∞ in $\mathscr{E}$.*

Remark. Actually for this theorem all we need is that $F \geqslant 0$ if $u \geqslant 0$. If further $F \leqslant 0$ when $u \leqslant 0$, the above argument gives a second continuum $\mathscr{C}_1{}^-$ in $\mathscr{P}^- \cup \{(\lambda_1, 0)\}$ which meets $(\lambda_1, 0)$ and ∞ in $\mathscr{P}^-$. Appropriate growth conditions on a_{ij}, b_i, etc. imply a priori bounds (depending on λ) for solutions of Eq. (2.8) (see [12]). This implies the projection of $\mathscr{C}_1{}^+$ on $\mathbf{R}$ contains (λ_1, ∞).

COROLLARY 2.13. *If F is independent of Du, Theorem* 2.12 *obtains for $\mathscr{C}_1{}^+$ and $\mathscr{C}_1{}^-$ without the positivity of F.*

Proof. The proof is the same as that of Theorem 2.12 modulo showing that $\mathscr{C}_1{}^\nu \subset \mathscr{P}^\nu \cup \{(\lambda_1, 0)\}$, $\nu = +, -$. We treat the $+$ case. If this is not the case, we can find $(\lambda, u) \in \mathscr{C}_1{}^+ \cap \mathbf{R}^+ \times \partial P^+$ where u has an interior zero in $\mathscr{D}$ or $\partial u/\partial\omega = 0$ at some boundary point of $\mathscr{D}$. Suppose $u(x_0) = 0$ at $x_0 \in \mathscr{D}$. Consider a small neighborhood Ω of x_0. It can be assumed that $|F(x, u, \lambda)| \leqslant a_0 |u|/2$ in Ω. Therefore $\mathscr{L}u \geqslant 0$ in Ω and the maximum principle implies $u \equiv 0$ in Ω and therefore in $\bar{\mathscr{D}}$ by a simple continuation argument. A similar argument works if $x_0 \in \partial\mathscr{D}$ and $\partial u(x_0)/\partial\omega = 0$. The proof continues as earlier.

Remark. In some work in progress, R.E.L. Turner has shown the existence of continua (in another sense) of solutions of a class of quasilinear elliptic equations.

3. Related Results

In this last section we shall show how some of the ideas developed in the previous sections can be used to prove the existence of continua of solutions for nonlinear eigenvalue problems where bifurcation need not occur. Let $\mathscr{E}$ and E be as earlier and let $\mathscr{E}^\nu = \{(\lambda, u) \in \mathscr{E} \mid \lambda \in \mathbf{R}^\nu\}$, $\nu = +, -$. Consider the equation

$$u = T(\lambda, u), \tag{3.1}$$

where $T : \mathscr{E} \to E$ is continuous and compact, and $T(0, u) \equiv 0$. It is not assumed that $T(\lambda, 0) \equiv 0$ and in fact the case of interest is when $T(\lambda, 0) \neq 0$ for $0 \neq \lambda$ near 0.

Note that (0, 0) is a solution of Eq. (3.1). Use of the Schauder fixed point theorem in a straightforward fashion shows that Eq. (3.1) possesses a solution $(\lambda, u(\lambda))$ for each $|\lambda|$ small. However a global result actually obtains here. Let $\mathscr{S}$ denote the set of solutions of Eq. (3.1).

THEOREM 3.2. *If T is continuous and compact on $\mathscr{E}$ and $T(0, u) \equiv 0$, then $\mathscr{S}$ contains a pair of continua $\mathscr{J}^+$, $\mathscr{J}^-$ lying in $\mathscr{E}^+$, $\mathscr{E}^-$, respectively, and meeting* (0, 0) *and* ∞.

Proof. If $\lambda_0 > 0$ is sufficiently small, the continuity of T implies that $T : [-\lambda_0, \lambda_0] \times \bar{B}_1 \to B_1$. Let $\Phi(\lambda, u) = u - T(\lambda, u)$. Then $d(\Phi(\lambda), B_1, 0) \equiv d(\Phi(\lambda), B_1)$ is well defined for $|\lambda| \leqslant \lambda_0$; and by the homotopy invariance of d,

$$d(\Phi(\lambda), B_1) \equiv \text{constant for} \quad |\lambda| \leqslant \lambda_0 . \tag{3.3}$$

For $\lambda = 0$, $\Phi(\lambda) = I$, the identity map on E. Hence

$$d(\Phi(\lambda), B_1) = d(I, B_1) = 1, \qquad |\lambda| \leqslant \lambda_0 . \tag{3.4}$$

Let $\mathscr{J}^\nu$ denote the (maximal) sub-continuum of $\mathscr{S}$ lying in $\mathscr{E}^\nu$, $\nu = +, -$, and which meets (0, 0). Since (0, 0) is the unique solution of Eq. (3.1) for $\lambda = 0$, $\mathscr{J}^+ \cap \mathscr{J}^- = (0, 0)$. If $\mathscr{J}^\nu$ does not meet infinity in $\mathscr{E}^\nu$, as in Lemma 1.2 a bounded open set $\mathcal{O}^\nu \subset \mathscr{E}^\nu$ can be found such that $\mathcal{O}^\nu \supset \mathscr{J}^\nu$ and $\mathcal{O}^\nu \cap \{0\} \times E \subset \{0\} \times B_1$. The argument used in the proof of Theorem 1.3 together with Eq. (3.4) shows that this is not possible. Thus the proof is complete.

An obvious corollary which shall not be formalized results when the solutions of Eq. (3.1) are a priori bounded as a function of λ. The projection of $\mathscr{J}^\nu$ on $\mathbf{R}^\nu$ is then $\mathbf{R}^\nu$, $\nu = +, -$.

If $T(\lambda, 0) \equiv 0$, Theorem 3.2 tells us nothing new since the trivial solutions $(\lambda, 0)$ are present.

To illustrate Theorem 3.2, consider the nonlinear integral equation

$$u(x) = \lambda \int_{\mathscr{D}} K(x, y) F(y, u(y), \lambda)\, dy \equiv T(\lambda, u), \tag{3.5}$$

where $x \in \bar{\mathscr{D}}$, a bounded domain in $\mathbf{R}^n$, $E = \mathscr{C}(\bar{\mathscr{D}})$, K is continuous on $\bar{\mathscr{D}} \times \bar{\mathscr{D}}$, and F is a continuous map of $\bar{\mathscr{D}} \times E \times R$ into E which is bounded on bounded sets. Then $T(\lambda, u)$ satisfies the hypotheses of Theorem 3.2 on $\mathscr{E}$ and there exists a pair of continua as in that theorem. If $| F(y, u, \lambda)| \leqslant M(\lambda)$ for all $y \in \bar{\mathscr{D}}$, $u \in E$, $\lambda \in \mathbf{R}$, with $M(\lambda)$ continuous on $\mathbf{R}$, then Eq. (3.5) possesses solutions for all $\lambda \in \mathbf{R}$.

A second example is provided by the quasilinear elliptic partial differential equation

$$\mathscr{L}u = Q(x, u, Du, \lambda) \quad \text{in } \mathscr{D} \qquad u = 0 \qquad \text{on} \quad \partial\mathscr{D}, \tag{3.6}$$

where $\mathscr{L}u$ is as in Eq. (2.8), the coefficients having the same properties as earlier. The function Q is assumed to be continuously differentiable in its arguments and $Q(x, u, p, 0) \equiv 0$. Let E be as earlier and define $T(\lambda, u)$ just as $G(\lambda, u)$ was defined before Eq. (2.9). Then Eq. (3.6) is equivalent to the equation $u = T(\lambda, u)$ in E with T satisfying the hypotheses of Theorem 3.2. Hence we obtain $\mathscr{S}^+, \mathscr{S}^-$. If $Q > 0$ for $\lambda > 0$ then the maximum principle implies that $T(\lambda, u) \subset P^+$ for $\lambda > 0$ and $\mathscr{S}^+ \subset \mathscr{P}^+ \cup \{(0, 0)\}$. Similarly, if $Q \geqslant 0$ for $\lambda > 0$, $T(\lambda, u) \subset \bar{P}^+$ for $\lambda > 0$ and $\mathscr{S}^+ \subset \bar{\mathscr{P}}^+ \cup (0, 0)$.

By formalizing the above, we have

THEOREM 3.7. *Equation* (3.6) *possesses a pair of continua of solution* $\mathscr{S}^+, \mathscr{S}^-$ *which meet* $(0, 0)$ *and infinity in* $\mathscr{E}^+, \mathscr{E}^-$, *respectively. If* $Q > 0$ $(\geqslant 0)$ *for* $\lambda > 0$, *then* $\mathscr{S}^+ \subset \mathscr{P}^+ \cup (0, 0)$ $(\bar{\mathscr{P}}^+ \cup (0, 0))$.

Remark. It is easily shown that these results are valid under more general boundary conditions. Some results related to Theorem 3.7 have been obtained by D. Cohen and H. Keller [5, 14] for the case in which the coefficients of $\mathscr{L}$ are independent of u, Du; $Q = Q(x, u) > 0$, and Q is monotonic increasing in u.

Next we investigate a situation related to but somewhat different from that of Eq. (3.1).

$$\mathscr{L}u = \mathscr{F}(\lambda, u), \tag{3.8}$$

where $\mathscr{L}$ is a linear map on E and $\mathscr{F}$ a nonlinear map of $\mathscr{E} \to E$ with $\mathscr{F}(0, u) \equiv 0$. If $\mathscr{L}^{-1}$ exists as a bounded map on E, then Eq. (3.8) can be put into the form (3.1). However, this may not be the case and, in particular, $\mathscr{L}$ may have a null space N. Despite this if Eq. (3.8) possesses sufficient structure, it may be still possible to convert it to an equation of the form (3.1) such that Theorem 3.2 can be used. Rather than develop a general theory here, we will restrict ourselves to one problem, viz., the question of the existence of time periodic solutions of a nonlinear wave equation.

Consider, then,

$$u_{tt} - u_{xx} = \lambda F(x, t, u) \qquad 0 < x < \pi, \quad 0 \leqslant t \leqslant 2\pi \tag{3.9}$$

together with the boundary and periodicity conditions:

$$\begin{aligned} u(0, t) = u(\pi, t) &= 0 \qquad 0 \leqslant t \leqslant 2\pi \\ u(x, t + 2\pi) &= u(x, t) \qquad 0 \leqslant x \leqslant \pi. \end{aligned} \tag{3.10}$$

The function F is assumed to be k ($\geqslant 3$) times continuously differentiable in its arguments, 2π periodic in t, and $\partial F/\partial t \not\equiv 0$. In addition, we assume F is strongly monotonic increasing in u, i.e.,

$$\partial F(x, t, u)/\partial u \geqslant \beta > 0 \quad \text{for all } x, t, u.$$

This problem was studied in [15] where the existence and local uniqueness of classical solutions of Eqs. (3.9) and (3.10) was shown for λ sufficiently small. By converting these two equations into an equation of the form (3.1), and verifying the conditions required for Theorem 3.2, we shall show that a global result actually obtains here. Free use shall be made of some of the preliminary results of [15].

First some notation. The completion of C^∞ functions in x, t on $[0, \pi] \times [0, 2\pi]$, 2π periodic in t, with respect to

$$| \varphi |_j^2 = \sum_{|\sigma| \leqslant j} \int_0^\pi \int_0^{2\pi} | D^\sigma \varphi |^2 \, dx \, dt,$$

will be denoted by H_j. (Here $D^\sigma \varphi$ denotes an arbitrary derivative of φ of order $| \sigma |$, the usual multi-index notation being employed.) Similarly $\mathring{H}_j$ denotes the completion as above of C^∞ function having support contained in $(0, \pi)$ with respect to x. H_j and $\mathring{H}_j$ are Hilbert spaces with respect to the inner product associated with $| \cdot |_j$. Let C_j denote the closure of C^∞ with respect to $\| \varphi \|_j = \sum_{|\sigma| \leqslant j} \max | D^\sigma \varphi |$.

The closure N in H_0 of the null space of the wave operator $\square \equiv \partial^2/\partial t^2 - \partial^2/\partial x^2$ under the boundary and periodicity conditions of Eq. (3.10) is the set of $\varphi \in H_0$ such that

$$\varphi(x, t) = p(x + t) - p(-x + t),$$

where p is 2π periodic and $\int_0^{2\pi} p^2(s)\, ds < \infty$. Let $N^\perp$ denote the orthogonal complement of N in H_0 and let P be the projector on N. Observe that N is infinite dimensional and if $\varphi \in N$ is smooth $(0, \varphi)$ is a classical solution of Eqs. (3.9) and (3.10). Thus any solutions of these equations with $\lambda \neq 0$ bifurcate from $(0, N)$.

Let $E = N^\perp \cap H_k$. It is shown in [15] that for each $w \in E$, there exists a unique $v = V(w) \in N \cap H_k$ such that $PF(\cdot, \cdot, v + w) = 0$, i.e.,

$$\int_0^\pi \int_0^{2\pi} F(x, t, v + w)\, \varphi\, dx\, dt = 0 \tag{3.11}$$

for all $\varphi \in N$. Theorems 1 and 3 of [15] further show that

$$| v |_k \leqslant c(w) \tag{3.12}$$

where $c(w)$ is a constant depending on bounds for $\| w \|_1$ and $| w |_k$. Since $k \geqslant 3$, bounds on $| w |_k$ imply corresponding bounds on $\| w \|_1$ via the Sobolev inequality [16]. Thus we can consider c as depending only on $| w |_k$.

The mapping $w \to V(w)$ is continuous as a map from $E \to N \cap H_{k-1}$ for if $(w_n) \subset E$ and $| w_n - w |_k \to 0$ as $n \to \infty$, then Eq. (3.12) gives uniform bounds for $(V(w_n))$ in $| \cdot |_k$. By the Sobolev inequality again, we get uniform bounds for $(V(w_n))$ in $\| \cdot \|_1$. Therefore by the Arzela–Ascoli theorem a subsequence of the $V(w_n)$ converges in C_0 to $\bar{V}$ satisfying

$$\int_0^\pi \int_0^{2\pi} F(x, t, \bar{V} + w)\, \varphi\, dx\, dt = 0, \tag{3.13}$$

for all $\varphi \in N$. The uniqueness of solutions V to Eq. (3.13) implies that $\bar{V} = V(w)$ and likewise that the entire sequence $V(w_n)$ converges to $V(w)$ in C_0. Taking $u = V(w_n) - V(w)$ and $p = k$ in the interpolation inequality [16],

$$| u |_j \leqslant c_1 | u |_0^{1-j/k} | u |_k^{j/k} \qquad 0 \leqslant j \leqslant k, \tag{3.14}$$

where c_1 is a constant and using the uniform boundedness of $(| V(w_n) - V(w)|_k)$ and the convergence of $V(w_n)$ to $V(w)$ in C_0, the continuity of $V(w)$ in $| \cdot |_{k-1}$ follows.

In [15] it is proved that if $f \in E$, there is a unique $w \in E \cap H_{k+1} \cap \mathring{H}_1$ such that $\square w = f$ and $| w |_{k+1} \leqslant c_2 | f |_k$, where c_2 is a constant. Therefore $\square^{-1}$ exists and by the Rellich theorem [17] is a compact map on E.

The "composition of functions" inequality implies that if $u \in H_k$, $F(x, t, u) \in H_k$ (see [15] or [18]). Combining these results, we see that if $w \in E \cap H_{k+1} \cap \mathring{H}_1$ and satisfies

$$w = \lambda \square^{-1} F(x, t, V(w) + w) \equiv T(\lambda, w), \tag{3.15}$$

then $V(w) + w$ is a classical solution of Eqs. (3.9) and (3.10) (see also [15]). Thus to solve Eqs. (3.9) and (3.10), it suffices to deal with Eq. (3.15) which is of the form (3.1).

If we show that $T(\lambda, w)$ is continuous and compact on $\mathscr{E}$, then Theorem 3.2 can be employed here. Since $V : E \to N \cap H_k$, and $F : V(w) + w \to E$ for $w \in E$ via the composition of functions inequality and the lemma cited above, the compactness of T follows from that of $\square^{-1} : E \to E \cap H_{k+1}$. To prove the continuity of T, let $(\lambda_n, w_n) \to (\lambda, w)$ in $\mathscr{E}$. As was shown above, $V(w_n) \to V(w)$ in C_0 and therefore $\lambda_n F(x, t, V(w_n) + w_n) \to \lambda F(x, t, V(w) + w)$ in C_0. Hence $\lambda_n \square^{-1} F(x, t, V(w_n) + w_n) = T(\lambda_n, w_n) \to T(\lambda, w)$ in H_1. By the interpolation inequality (3.14) with $p = k + 1$, $j = k$, and $u = T(\lambda_n, w_n) - T(\lambda, w)$, it follows that $T(\lambda_n, w_n) \to T(\lambda, w)$ in $|\cdot|_k$.

Thus Theorem 3.2 is applicable here. Since the map $w \to V(w) + w$, $E \to H_{k-1} \cap \mathring{H}_1$ is continuous, we have proved the global result:

THEOREM 3.16. *There exist a pair of continua of solutions of Eqs.* (3.9) *and* (3.10) *in* $H_{k-1} \cap \mathring{H}_1$ *which meet* $(0, V(0))$ *and* ∞.

REFERENCES

1. M. A. KRASNOSELSKI, "Topological Methods in the Theory of Nonlinear Integral Equations," Chap. 2, Macmillan, New York, 1965.
2. J. LERAY AND J. SCHAUDER, Topologie et équations fonctionelles, *Ann. Sci. École Norm.* Suppl. 3, 51 (1934), 45–78.
3. P. H. RABINOWITZ, Nonlinear Sturm–Liouville problems for second order ordinary differential equations, *Comm. Pure Appl. Math.*, to appear.
4. M. G. CRANDALL AND P. H. RABINOWITZ, Nonlinear Sturm–Liouville eigenvalue problems and topological degree, *J. Math. Mech.*, to appear.
5. J. B. KELLER AND S. ANTMAN (Eds.), "Bifurcation Theory and Nonlinear Eigenvalue Problems," Benjamin, New York, 1969.
6. G. T. WHYBURN, "Topological Analysis," Chap. 1, Princeton Univ. Press, Princeton, N. J., 1958.
7. L. BAUER, E. L. RIESS, AND H. B. KELLER, "Antisymmetric Buckling of Hollow Spheres and Hemispheres," preprint.
8. M. G. CRANDALL AND P. H. RABINOWITZ, Bifurcation at simple eigenvalues, *J. Functional Analysis*, to appear.

9. F. R. Gantmacher and M. G. Krein, "Oszillationsmatrizen, Oszillationkerne und kleine Schwingungen mechanischer Systeme," Academie-Verlag, Berlin, 1960.
10. R. Turner, Nonlinear eigenvalue problems with nonlocal operators, *J. Math. Mech.*, to appear.
11. M. G. Krein and M. A. Rutman, Linear operators which leave a cone in Banach space invariant, *Amer. Math. Soc. Trans. Ser. 1*, **10** (1950).
12. O. A. Ladyzhenskaya and N. Uraltseva, "Linear and Quasilinear Elliptic Equations," Chap. 6, Academic Press, New York/London, 1968.
13. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. II, Chap. 4, Interscience, New York, 1962.
14. H. B. Keller and D. S. Cohen, Some positone problems suggested by nonlinear heat generation, *J. Math. Mech.* **16** (1967), 1361–1376.
15. P. H. Rabinowitz, Periodic solutions of nonlinear hyperbolic partial differential equations, *Comm. Pure Appl. Math.* **20** (1967), 145–205.
16. L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola, Norm. Super. Pisa Ser. 3* **13** (1959), 1–48.
17. L. Bers, F. John, and M. Schechter, "Partial Differential Equations," Chap. 4, Interscience, New York, 1964.
18. J. Moser, A rapidly convergent iteration method and nonlinear partial differential equations, *Ann. Scuola, Norm. Super. Pisa Ser. 3* **20** (1966), 265–315.
19. S. Karlin, The existence of eigenvalues for integral operators, *Trans. Amer. Math. Soc.* **113** (1964), 1–17.
20. S. V. Parter, "Nonlinear Eigenvalue Problems for Some Fourth-Order Equations: II. Fixed Point Methods," Technical report #77, pp. 1–46, Computer Sci. Dept., Univ. of Wisconsin, Nov. 1969.