Reaction-Diffusion Models and Bifurcation Theory
Lecture 2: Models in growth and interaction

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Mathematical Models

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_Albert Einstein_

All models are wrong, but some are useful.  

_George Box_
Growth of a single species: unbounded

\[ y' = yf(y), \quad f(y) \text{ is the growth rate per capita} \]
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growth rate \( yf(y) = y^q \)

\( q < 0 \): sublinear growth (concave);
\( q = 0 \): linear growth;
\( 0 < q < 1 \): superlinear polynomial growth
\( q = 1 \): superlinear exponential growth (so far all globally exist)
\( q > 1 \): superlinear hyperbolic growth (blow up in finite time)
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\[ y(t) = \frac{y_0 e^{kt}}{1 + y_0 N^{-1}(e^{kt} - 1)}, \quad |y(t) - N| \sim e^{-kt} \quad (t \to \infty) \]
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\[ y(t) = \frac{y_0 e^{kt}}{[1 + y_0^q N^{-q}(e^{kqt} - 1)]^{1/q}}, \quad |y(t) - N| \sim e^{-kqt} \quad (t \to \infty) \]
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(Gompertz) \[ y' = k(\log N - \log y)y \quad [\text{Gompertz, 1832}] \quad f(y) = k(\log N - \log y) \]

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(Holling) \[ y' = \frac{ky}{h^p + y^p} \quad (h > 0, \ p > 0) \quad (f(y) = \frac{k}{h^p + y^p}) \]
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Common character: \[ yf(y) \] has a local maximum (one-hump), and \[ f(y) \] is decreasing solution curve is increasing and convex-concave
Sigmoid functions

We define a sigmoid function to be $y : (-\infty, \infty) \to (0, \infty)$ such that $y'(t) > 0$, $y''(t)(t - t_0) < 0$ for all $t \neq t_0$, $\lim_{t \to -\infty} y(t) = 0$. In general any solution of $y' = yf(y)$ with decreasing $f(y)$ is a sigmoid function. In many cases, $\lim_{t \to \infty} y(t) = N$ (the carrying capacity when $f(N) = 0$), and it is also called a generalized logistic curve. One can choose $f(y)$ to fit the data to an appropriate sigmoid function.

Typical sigmoid functions:

- $u(t) = \frac{1}{1 + ae^{-k(t-t_0)}}$ ($t = t_0$ is the inflection point where $u''(t_0) = 0$)
- $u(t) = \frac{1}{(1 + ae^{-k(t-t_0)})^q}$ ($t = t_0$ is the inflection point where $u''(t_0) = 0$)
- $u(t) = 0.5[\tanh(t) + 1] = \frac{0.5}{1 + e^{-2t}}$
- $u(t) = 0.5[\text{erf}(t) + 1] = 0.5 \left[ \frac{2}{\sqrt{\pi}} \int_{-\infty}^{t} e^{-s^2} ds + 1 \right]$
- $u(t) = \frac{0.5(t + 1)}{\sqrt{t^2 + 1}}$; $u(t) = 0.5 \left[ \frac{\pi}{2} \arctan(t) + 1 \right]$

In many other problems, sigmoid functions have range $(-1, 1)$ instead of $(0, 1)$.
Sigmoid functions

\[
\begin{align*}
\text{erf}\left( x \right) &= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} dt \\
\frac{x}{\sqrt{1 + x^2}} &= \frac{\text{erf}\left( \frac{\sqrt{\pi}}{2} x \right)}{\sqrt{1 + x^2}} \\
\tanh(x) &= \frac{2}{\pi} \arctan\left( \frac{\pi}{2} x \right) \\
\frac{2}{\pi} \text{gd}(\frac{\pi}{2} x) &= \frac{x}{1 + |x|}
\end{align*}
\]
Hill function

Hill function [Hill, 1919] is a function in form $f(u) = \frac{u^p}{h^p + u^p}$, for $u \geq 0$, $p > 0$. The point $u = h$ is the half-saturation point since $f(h) = 1/2$, the half of the asymptotic limit. When $p > 1$, $u = h$ is also the inflection point where $f''(u) = 0$. When $p = \infty$, the limit is the step function $f(u) = 0$ for $u < h$ and $f(u) = 1$ for $u > h$. 
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One can define a generalized Hill function by $f : [0, \infty) \to [0, 1)$ such that $f(0) = 0$, $f'(u) > 0$ for $u > 0$, and $\lim_{u \to \infty} f(u) = 1$. One can obtain a Hill function by restricting and shifting a sigmoid function.
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In ecology, people use the Hill function as the predator functional response. [Holling 1959] $p = 1$ is the type II, and $p > 1$ is the type III.
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In biochemistry, people use the Hill function as the reaction rate. When $p = 1$, it is called Michaelis-Menton chemical kinetics. [Menten and Michaelis, 1913]
Hill functions
Growth of a single species: Allee effect

(strong Allee effect) \[ y' = ky \left(1 - \frac{y}{N}\right) \left(\frac{y}{M} - 1\right) \] \[\text{[Allee, 1931]}\] \[0 < M < N\]

\[f(y) = k \left(1 - \frac{y}{N}\right) \left(\frac{y}{M} - 1\right)\]
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(weak Allee effect) \( y' = ky \left(1 - \frac{y}{N}\right) \left(\frac{y}{M} + 1\right) \) \text{[Allee, 1931]} 0 < M < N

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(degenerate logistic, weak Allee effect) \( y' = ky^p \left( 1 - \frac{y}{N} \right) \) \( (p > 1) \) (autocatalytic chemical reaction)
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(another weak Allee effect function) \( y' = ky \left( 1 - \frac{y}{N} \right) \left( \frac{y}{y + M} \right) \), \( M, N > 0 \)

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(another weak Allee effect function) \( y' = ky \left(1 - \frac{y}{N}\right) \left(\frac{y}{y + M}\right) \), \( M, N > 0 \)

\( f(y) = k \left(1 - \frac{y}{N}\right) \left(\frac{y}{y + M}\right) \)

Common character: \( yf(y) \) has a local maximum and may have a local minimum, and \( f(y) \) is increasing then decreasing. The solutions are also sigmoid functions.
Ecology interactions

[Lotka, 1920], [Volterra, 1926] $x' = ax - bxy$, $y' = -cy + dxy$

[Volterra, 1926] $x' = x(a - bx - cy)$, $y' = y(d - ey \pm fx)$

[Gause 1934] $x' = xg(x) - yp(x)$, $y' = yq(x) - \gamma y$

[Kolmogorov, 1936] $x' = xf(x, y)$, $y' = yg(x, y)$, $\frac{\partial f}{\partial y}(x, y) < 0$ and $\frac{\partial g}{\partial x}(x, y) > 0$. 
Predator-prey Models

Rosenzweig-MacArthur type [Rosenzweig-MacArthur, 1963]
\[ x' = xf(x) - \phi(x, y)y, \quad y' = yg(y) + k\phi(x, y)y \]

\( f(x) \): growth rate per capita of the prey, \( g(y) \): growth (or death) rate of the predator

\( \phi(x, y) \): predator functional response

Holling type: \( \phi(x) = \frac{ax^p}{h^p + x^p} \quad (p \geq 1) \) (monotone increasing)

Non-monotone: \( \phi(x) = \frac{ax}{b + cx + x^p} \quad (p \geq 1) \)

Beddington-DeAngelis: \( \phi(x, y) = \frac{ax}{b + cx + dy} \)

Ratio-dependent: \( \phi(x, y) = \frac{ax}{cx + dy} = \frac{a}{c + dy/x} \)

[Leslie-Gower, 1960] \( u' = u(a - bu - cv), \quad v' = v \left( d - \frac{ev}{u} \right) \)

[Tanner, 1975] \( u' = u \left( a - bu - \frac{cv}{m + u} \right), \quad v' = v \left( d - \frac{ev}{u} \right) \)

Name: predator-prey, consumer-resource, activator-inhibitor

Competitive and cooperative systems
(Bio)chemical reaction

\[ mA + nB \xrightarrow{k_1} pC + qD \]
(Bio)chemical reaction

$mA + nB \xrightarrow{k_1} pC + qD$

Reaction rate $R = -\frac{1}{m} \frac{dA}{dt} = -\frac{1}{n} \frac{dB}{dt} = \frac{1}{p} \frac{dC}{dt} = \frac{1}{q} \frac{dD}{dt}$
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**Law of mass action**: the reaction rate is proportional to the product of the concentrations of the participating molecules so \( R = k_1 A^m B^n \), \( (m + n \text{ reaction order}) \)
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**Example:** Lotka reaction [Lotka, 1920]

\[ A + X \xrightarrow{k_1} 2X, \quad X + Y \xrightarrow{k_2} 2Y, \quad Y \xrightarrow{k_3} P \]

where \( A \) is the grass, \( X \) is the rabbits, \( Y \) is the foxes, and \( P \) is the dead foxes. Assuming that \( A \) is a constant (infinite amount of grass).

\[ X' = k_1 AX - k_2 XY, \quad Y' = k_2 XY - k_3 Y \]
Reversible chemical reaction

\[ mA + nB \xrightleftharpoons[k_2]{k_1} pC + qD \]
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\(R = k_1 A^m B^n - k_2 C^p D^q,\)

\(A' = -m(k_1 A^m B^n + k_2 C^p D^q),\ B' = -n(k_1 A^m B^n + k_2 C^p D^q),\)

\(C' = p(k_1 A^m B^n + k_2 C^p D^q),\ D' = q(k_1 A^m B^n + k_2 C^p D^q).\)
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**Example:** single molecule reaction \( A \xrightleftharpoons[k_2]{k_1} B \)

\[ A' = -k_1 A + k_2 B, \quad B' = k_1 A - k_2 B. \]

Solution: \( A(t) + B(t) = A_0 + B_0 \), then

\[ A' = -k_1 A + k_2 (A_0 + B_0 - A) = k_2 (A_0 + B_0) - (k_1 + k_2)A \]

So \( \lim_{t \to \infty} A(t) = \frac{k_2}{k_1 + k_2} (A_0 + B_0) \), and \( \lim_{t \to \infty} B(t) = \frac{k_1}{k_1 + k_2} (A_0 + B_0) \).

The reaction reaches an equilibrium asymptotically. In equilibrium, \( \frac{A_\infty}{B_\infty} = \frac{k_2}{k_1} \).
Autocatalytic reaction

A chemical reaction is autocatalytic if the reaction product itself is the catalyst for that reaction.

\[ mA + nB \xrightarrow{k_1} (n+p)B \quad A' = -mk_1 A^m B^n, \quad B' = pk_1 A^m B^n \]

\[ mA + nB \xrightarrow{k_1 \quad k_2} (n+p)B \]

\[ A' = -mk_1 A^m B^n + mk_2 B^{n+p}, \quad B' = pk_1 A^m B^n - pk_2 B^{n+p}. \]

**Examples:** Lotka reaction

[Schnakenberg, 1979] \[ 2X + Y \rightarrow 3X, \quad A \rightarrow Y, \quad X \xrightleftharpoons{} B \]

\[ x' = x^2 y - x + b, \quad y' = -x^2 y + a. \]

Brusselator [Prigogine, 1980]

\[ A \rightarrow X, \quad 2X + Y \rightarrow 3X, \quad B + X \rightarrow Y + D, \quad X \rightarrow E \]

\[ x' = a + x^2 y - (b + 1)x, \quad y' = bx - x^2 y. \]

[Gray-Scott, 1983] \[ U + 2V \rightarrow 3V, \quad V \xrightarrow{k} P, \text{ and feeding } U \text{ (amount } F), \text{ removing } U \text{ and } V \text{ (amount } F) \]

\[ U' = -UV^2 + F(1 - U), \quad V' = UV^2 - (F + k)V. \]
Simple autocatalytic reactions

\[ A + nX \xrightleftharpoons{\frac{k_1}{k_2}} (n + 1)X \]
Simple autocatalytic reactions

\[ A + nX \xrightleftharpoons[k_1]{k_2} (n + 1)X \]

Mass action kinetics: \( x(t) \) = amount of \( X \) molecules, \( a(t) \) = amount of \( A \) molecules

\[ \frac{dx}{dt} = k_1 x^n a - k_2 x^{n+1}, \quad \frac{da}{dt} = -k_1 x^n a + k_2 x^{n+1}. \]
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\( x(t) + a(t) \) = constant, assumed to be 1. Then

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\]

Assume \( k_1 = k_2 = 1 \).

\(n = 0: \text{Unimolecular reaction } A \underset{\rightleftharpoons}{\xrightarrow{k_1/k_2}} X \text{ (uncatalyzed)}\)

\[
\frac{dx}{dt} = 1 - 2x, \quad \text{solution} \quad x(t) = 0.5 + (x_0 - 0.5) e^{-2t}.
\]
Solution of autocatalytic equations

\( n = 1 \): Bimolecular reaction \( A + X \rightleftharpoons 2X \) (first order) (logistic equation!)

\[
\frac{dx}{dt} = x - 2x^2, \quad \text{solution} \quad x(t) = \frac{x_0}{2x_0 + (1 - 2x_0)e^{-t}}.
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\]

\( n = 2 \): Trimolecular reaction \( A + 2X \rightleftharpoons 3X \) (second order)

\[
\frac{dx}{dt} = x^2 - 2x^3, \quad \text{solution} \quad t = 2 \ln \left( \frac{x(1 - 2x_0)}{x_0(1 - 2x)} \right) + \frac{1}{x_0} - \frac{1}{x}.
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\]

\( n = 2 \): Trimolecular reaction \( A + 2X \iff 3X \) (second order)

\[
\frac{dx}{dt} = x^2 - 2x^3, \quad \text{solution} \quad t = 2 \ln \left( \frac{x(1 - 2x_0)}{x_0(1 - 2x)} \right) + \frac{1}{x_0} - \frac{1}{x}.
\]

Red: \( n = 0 \); Blue: \( n = 1 \); Green: \( n = 2 \)  
\( k_1 = k_2 = 1 \), \( x_0 = 0.05 \)
Solution of autocatalytic equations

Red: \( n = 0 \); Blue: \( n = 1 \); Green: \( n = 2 \)

\( k_2 = 0, \ x_0 = 0.01, \ k_1 \) are different for three curves and \( k_1 \) are chosen so \( x(5) = 0.5 \)

The solutions for \( n \geq 1 \) are all sigmoid function with limits 0 and 1 at \( t = -\infty \) and \( t = \infty \).
Nonlinear reaction rates: Michaelis-Menten

In a basic enzyme reaction, an enzyme $E$ binds to a substrate $S$ to form a complex $ES$, which in turn is converted into a product $P$ and the enzyme $E$. (Enzyme is a biological catalyst.)

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1. $v + w$ is a constant, and assume it is $E_0$.
2. (Quasi steady state approximation) Assume $v' = 0$ and $w' = 0$. Then

$$-u' = w' = \frac{E_0 k_1 u}{k_2 + k_3 + k_1 u}$$
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So in a large network of chemical reactions, one may use $S \rightarrow P$ to replace the three equations above, but the reaction rate is the Michaelis-Menten function.
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There are many more complex enzyme reactions which make chemical reaction network (large system of ODEs satisfying mass action law and other laws).
Modeling for the CIMA reaction


\[ MA + I_2 \rightarrow IMA + I^- + H^+ , \]
\[ ClO_2 + I^- \rightarrow ClO_2 + (1/2)I_2 , \]
\[ ClO_2 + 4I^- + 4H^+ \rightarrow Cl^- + 2I_2 + 2H_2O \]

\([ClO_2], [I_2]\) and \([MA]\) varying slowly, assumed to be constant

Let \( I^- = X, ClO_2 = Y \) and \( I_2 = A \). Then the reaction becomes

\[ A \xrightarrow{k_1} X, X \xrightarrow{k_2} Y, 4X + Y \xrightarrow{k_3} P \]

Reaction rates \( k_1, k_2 \) are constants, and \( k_3 \) is proportional to \( \frac{[X] \cdot [Y]}{u + [X]^2} \)

\[ u' = a - u - \frac{4uv}{1 + u^2} , \quad v' = b(u - \frac{uv}{1 + u^2}). \]

Here \([I^-] = u(t), [ClO_2] = v(t), a, b > 0\). Key parameter: \( a > 0 \) (the feeding rate)
Activator-inhibitor systems

In the law of mass action, a gain is represented by a $+\,\text{term}$, and a loss is represented by a $-\,\text{term}$. In more general case, if the presence of one chemical $u$ increases the generation of another chemical $v$, then $u$ is called a activator, and if $u$ decreases the generation of another chemical $v$, then $u$ is an inhibitor. If $u$ is an activator of $v$, and $v$ is an inhibitor of $u$, then in
$$u' = f(u, v), \quad v' = g(u, v),$$
$f_v < 0$ and $g_u > 0$.

[Gierer-Meinhardt, 1972]

$$a' = \rho \frac{a^2}{h} - \mu_a a + \rho_a, \quad h' = \rho a^2 - \mu_h h + \rho_h.$$

Here $a$ is the activator, and $h$ is the inhibitor. $\mu_a$ and $\mu_h$ are decay rates of $a$ and $h$, $\rho_a$ and $\rho_h$ are the feeding rates of $a$ and $h$, and $\rho$ is the reaction rate.
More models

Water-plant model
\[ w' = a - w - wn^2, \quad n' = wn^2 - mn \]
\( w(t) \) water, \( n(t) \) plant

\( a > 0 \): rainfall; \( -w \): evaporation; \( -wn^2 \): water uptake by plants; \( wn^2 \): plant growth; \( -mn \): plant loss

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\(-mn\): plant loss

[Klausmeier, Science, 1999]

Epidemic models
\[ S' = A - \alpha SI - AS, \quad I' = \alpha SI - \beta I - AI, \quad R' = \beta I - AR \]
assuming that \(S(0) + I(0) + R(0) = 1\)

[Kermack and McKendrick, 1927]
Pattern of autocatalytic models

\[ 2U + V \rightarrow 3U, \quad U \rightarrow P \]

Brusseller \( A \rightarrow U, \quad B + U \rightarrow V \)
\[ u' = u^2v - u + a - bu, \quad v' = -u^2v + bu \]

Schnakenberg \( A \rightarrow U, \quad B \rightarrow V \)
\[ u' = u^2v - u + a, \quad v' = -u^2v + bu \]

Klausmeier \( V + C \rightarrow Q, \quad B \rightarrow V \)
\[ u' = u^2v - u, \quad v' = -u^2v + b - cv \]

Gierer-Meinhardt \( A \rightarrow U, \quad B \rightarrow V, \quad V + C \rightarrow Q \)
\[ u' = u^2v - u + a, \quad v' = -u^2v + b - cv \]

Some systems have multiple equilibria, and some systems have limit cycles.

**Conjecture:**
1. For system \( u' = u^2v + au + bv + c, \quad v = -u^2v + du + ev + f \), there is at most one periodic orbit;
2. For system \( u' = uv + au + bv + c, \quad v = -uv + du + ev + f \), there is no periodic orbit.
Patterns in ODE

For scalar autonomous ODEs, each solution is monotone, so it is likely a sigmoid function. Then the only “pattern” is how it approaches an equilibrium.
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Self-organization is a process where some form of global order or coordination arises out of the local interactions between the components of an initially disordered system. This process is spontaneous: it is not directed or controlled by any agent or subsystem inside or outside of the system; however, the laws followed by the process and its initial conditions may have been chosen or caused by an agent. It is often triggered by random fluctuations that are amplified by positive feedback.
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The Tao produced One; One produced Two; Two produced Three; Three produced All things. All things leave behind them the Obscurity (out of which they have come), and go forward to embrace the Brightness (into which they have emerged), while they are harmonised by the Breath of Vacancy.—– Laozi (Lao-Tse) ‘Tao Te Ching’ Chapter 42
Van der Pol equation

\[ x'' + k(x^2 - 1)x' + x = 0, \quad \text{or} \quad x' = y, \ y' = -k(x^2 - 1)y - x. \]

\( k = 0: \ x'' + x = 0 \) (harmonic oscillator)

Left: phase portrait; Right: time series
Van der Pol equation

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\( k = 0.1: \ x'' + 0.1(x^2 - 1)x' + x = 0 \) (limit cycle with amplitude 2)

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Van der Pol equation

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\( k = 2: \) \( x'' + 2(x^2 - 1)x' + x = 0 \) (limit cycle with amplitude 2, relaxation oscillator)

Left: phase portrait; Right: time series
Van der Pol relaxation oscillator

Left: $k = 0.1$; Middle: $k = 2$; Right: $k = 10$. 
FitzHugh-Nagumo oscillator

[\textit{FitzHugh, 1961}, \textit{[Nagumo, et.al., 1961]} (neuron conduction model)]

\[v' = -v(v - a)(v - 1) - w + I, \quad w' = bv - cw.\]

\[a = 0.25, \quad b = c = 0.002, \quad I = 0.4167.\]

Left: phase portrait; Right: time series
Relaxation oscillation in predator-prey model


Relaxation oscillator profile of limit cycle in predator-prey system.

\[
\begin{align*}
\frac{du}{dt} &= u (1 - u) - \frac{muv}{a + u}, \\
\frac{dv}{dt} &= -dv + \frac{muv}{a + u}
\end{align*}
\]

Parameters: \(a = 0.5, m = 1, d = 0.1, \lambda = 1/18 \approx 0.056\), period \(T \approx 37\).
Small $d$

Parameters: $a = 0.5$, $m = 1$, $d = 0.01$, $\lambda = 1/198 \approx 0.005$, period $T \approx 336$.
Lorenz’s butterfly

In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection. The model is a system of three ordinary differential equations now known as the Lorenz equations:

\[
\begin{align*}
\frac{dx}{dt} &= \sigma (y - x), \\
\frac{dy}{dt} &= x(\rho - z) - y, \\
\frac{dz}{dt} &= xy - \beta z.
\end{align*}
\]

Here \(x\), \(y\), and \(z\) make up the system state, \(t\) is time, and \(\sigma, \rho, \beta\) are the system parameters.
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Here \(x\), \(y\), and \(z\) make up the system state, \(t\) is time, and \(\sigma, \rho, \beta\) are the system parameters.

Take \(\sigma = 10\), \(\beta = 8/3\) and \(\rho = 28\).

Left: 3D phase portrait; Middle: \(x - z\) phase portrait; Right: Time series for \(y(t)\).
Rössler half-twisted Möbius strip

Otto Rössler designed the Rössler attractor in 1976. The Rössler attractor was intended to behave similarly to the Lorenz attractor, but also be easier to analyze qualitatively.

\[
\begin{align*}
\frac{dx}{dt} &= -y - z, \\
\frac{dy}{dt} &= x + ay, \\
\frac{dz}{dt} &= b + z(x - c).
\end{align*}
\]

Take \(a = 0.2, \ b = 0.2, \ c = 5.7\).

Left: 3D phase portrait; Middle: time series for \(x(t)\); Right: Rössler attractor.
Chua’s double scroll

[Chua, 1983] The Chua Circuit was invented in the fall of 1983 in response to two unfulfilled quests among many researchers on chaos: 1. devise a laboratory system which can be realistically modeled by the Lorenz Equations; 2. prove that the Lorenz attractor, which was obtained by computer simulation, is indeed chaotic in a rigorous mathematical sense.

\[
\begin{align*}
\frac{dx}{dt} &= \alpha(y + cx - x^3), \\
\frac{dy}{dt} &= x - y + z, \\
\frac{dz}{dt} &= -\beta y.
\end{align*}
\]

Take \( \alpha = 10 \), \( \beta = 16 \) and \( c = 0.143 \).

Left: 3D phase portrait; Right: time series for \( x(t) \).

Leon Chua (the inventor of Chua Circuit), is the dad of “tiger mom” Amy Chua (Yale law professor), who wrote “Battle Hymn of the Tiger Mother” (2011).
Hastings-Powell’s teapot in food-chain model

[Hastings-Powell, 1991, Ecology]: Chaos in a three species food chain

\[
x' = x(1 - x) - \frac{a_1 xy}{1 + b_1 x}, \quad y' = \frac{a_1 xy}{1 + b_1 x} - d_1 y - \frac{a_2 yz}{1 + b_2 y}, \quad z' = \frac{a_2 yz}{1 + b_2 y} - d_2 z.
\]

Take \(a_1 = 5, a_2 = 0.1, b_1 = 4, b_2 = 2, d_1 = 0.4, d_2 = 0.01\).

Left: 3D phase portrait; Middle: \(x - y\) phase portrait; Right: Time series for \(x(t)\) and \(y(t)\).
Biochemical feedback control circuits

\[ \frac{dx_1}{dt} = g(x_n) - \alpha_1 x_1, \quad x_i' = x_{i-1} - \alpha_i x_i, \quad 2 \leq i \leq n \]

where \( \alpha_i > 0 \) and the feedback function \( g(u) \) is a bounded continuously differentiable function satisfying

\[ 0 < g(u) < M, \quad g'(u) > 0, \quad u > 0. \]

Hence it models a positive feedback. For example \( g(x_n) = \frac{1 + x_n^p}{K + x_n^p} \).

When there are exactly three equilibria, the smallest and the largest are stable, and there is a threshold manifold separating the basins of attraction. For \( n \leq 3 \), all orbits tend to the corresponding saddle point on threshold manifolds; for \( n \geq 5 \), there may exist Hopf bifurcation on the unique threshold manifold. But for \( n = 4 \), whether there is a nontrivial periodic orbit or not on threshold manifold is an open problem.

[Jiang-Shi, 2009]
Hypercycle


\[ x'_i = x_i \left( x_{i-1} - \sum x_j x_{j-1} \right), \quad i = 1, 2 \cdots, n, \ (\text{mod } n) \]


Hypercycle Dynamics

(left): $n = 3$, (middle): $n = 4$, (right): $n = 5$

Mathematical result:

[Hofbauer-Mallet-Paret-Smith, 1991, JDDE] When $n \leq 4$, all solutions converge to the steady state; and when $n \geq 5$, all solutions converge to a periodic orbit.
Five Elements Theory

According to ancient Chinese philosophy, the five elements are **Metal**, **Wood**, **Water**, **Fire**, and **Earth**. The system of five elements was used for describing interactions and relationships between phenomena. It was employed as a device in many fields of early Chinese thought, including seemingly disparate fields such as geomancy or Feng shui, astrology, traditional Chinese medicine, music, military strategy and martial arts.

The doctrine of five phases describes two cycles, a generating or creation (**Sheng**) cycle, and an overcoming or destruction (**Ke**) cycle. The **Sheng** cycle is Wood feeds Fire; Fire creates Earth; Earth bears Metal; Metal carries Water; Water nourishes Wood. And the **Ke** cycle is Wood parts Earth; Metal chops Wood; Fire melts Metal; Water quenches Fire; Earth dams Water.
Five Elements in western world

![Diagram of the Five Elements]

- Rock covers Paper and Paper covers Rock
- Rock crushes Scissors and Scissors crushes Rock
- Paper cuts Lizard and Lizard eats Paper
- Lizard poisons Spock and Spock poisons Lizard
- Scissors smashes Rock and Rock smashes Scissors

![Photo of two men discussing]

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**Conclusion**

- The Five Elements provide a unique perspective on interactions and transformations in the western world.
Five Elements in Western World

Five Elements

Growth
Reaction
Oscillations
3D
Higher D

Conclusion

Five Elements

![Diagram of Five Elements]

![Images of TV Show and Movie Posters]
Initial modeling

[Xiaoling Wang, Yuwen Wang, Junping Shi, 2011]

\[
\begin{align*}
    x_1' &= x_1(x_5 - x_4), \\
    x_2' &= x_2(x_1 - x_5), \\
    x_3' &= x_3(x_2 - x_1), \\
    x_4' &= x_4(x_3 - x_2), \\
    x_5' &= x_5(x_4 - x_3).
\end{align*}
\]

Same equations can be written for $n = 3, 4$ or $n \geq 6$. 
Initial modeling

[Xiaoling Wang, Yuwen Wang, Junping Shi, 2011]

\[
\begin{align*}
    x'_1 &= x_1(x_5 - x_4), \\
    x'_2 &= x_2(x_1 - x_5), \\
    x'_3 &= x_3(x_2 - x_1), \\
    x'_4 &= x_4(x_3 - x_2), \\
    x'_5 &= x_5(x_4 - x_3).
\end{align*}
\]

Same equations can be written for \( n = 3, 4 \) or \( n \geq 6 \).

(left): \( n = 3 \); (middle): \( n = 4 \); (right): \( n = 5 \)
when \( n = 3 \), every solution is a periodic orbit;
when \( n \geq 4 \), the solution blows up at a finite time.
Second attempt

\[
\begin{align*}
x'_1 &= x_1(x_5 - x_1x_4), \\
x'_2 &= x_2(x_1 - x_2x_5), \\
x'_3 &= x_3(x_2 - x_3x_1), \\
x'_4 &= x_4(x_3 - x_4x_2), \\
x'_5 &= x_5(x_4 - x_5x_3).
\end{align*}
\]

(left): \(0 \leq t \leq 20\), (right): \(0 \leq t \leq 50\).
Third Attempt

\[ \begin{align*}
    x'_1 &= x_1(x_5 - x_1 x_4 - a x_2), \\
    x'_2 &= x_2(x_1 - x_2 x_5 - a x_3), \\
    x'_3 &= x_3(x_2 - x_3 x_1 - a x_4), \\
    x'_4 &= x_4(x_3 - x_4 x_2 - a x_5), \\
    x'_5 &= x_5(x_4 - x_5 x_3 - a x_1).
\end{align*} \]

(Left): \(0 \leq t \leq 50\), (Right): \(0 \leq t \leq 100\).
For differential equations with one or two variables, the only patterns are equilibrium and limit cycle. In 1D, the solution can approach to the equilibrium in different time scale; in 2D, the limit cycle can have many different profiles, which shows the character of the physical phenomenon described by the model. In 3D or higher, the dynamics can be chaotic in general, but some oscillatory patterns and associated strange attractors still emerge. Many problems still await further studies.

Higher dimensional systems of differential equations can be used in many fundamental scientific theories, such as Darwin evolution theory, self-organization of life, and other social science questions. Many such systems may contain certain degree of symmetries, which can generate special patterns.

Differential equations on a network (a graph) are even more difficult to study.