Abstract local and global bifurcation theory of steady state problems

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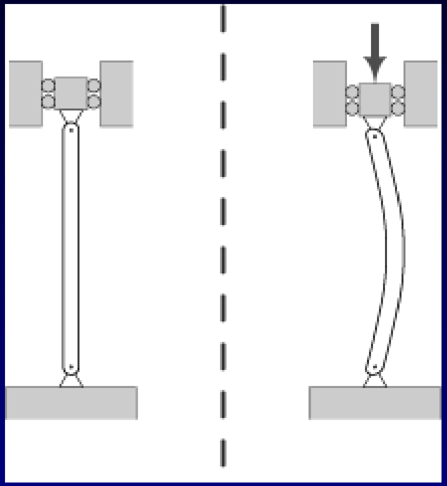
Earliest Bifurcation: Euler buckling



In engineering, buckling is a failure mode characterized by a sudden failure of a structural member subjected to high compressive stresses, where the actual compressive stresses at failure are greater than the ultimate compressive stresses that the material is capable of withstanding. This mode of failure is also described as failure due to elastic instability.

$\phi'' + \lambda \sin\phi = 0, x \in (0, \pi), \phi(0) = \phi(\pi) = 0$

[Da Vinci, 1452-1519] [Euler, 1750]

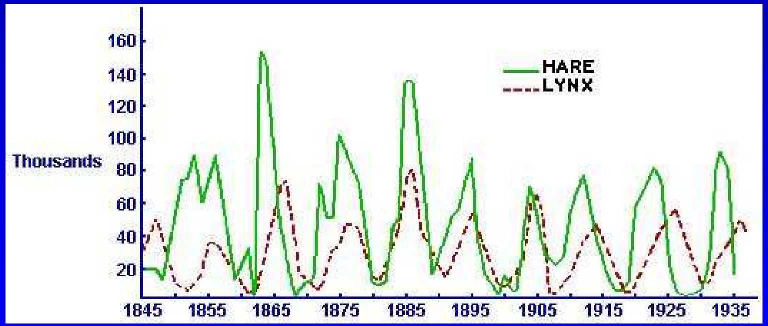
Bifurcation in predator-prey model

Predator-prey interaction

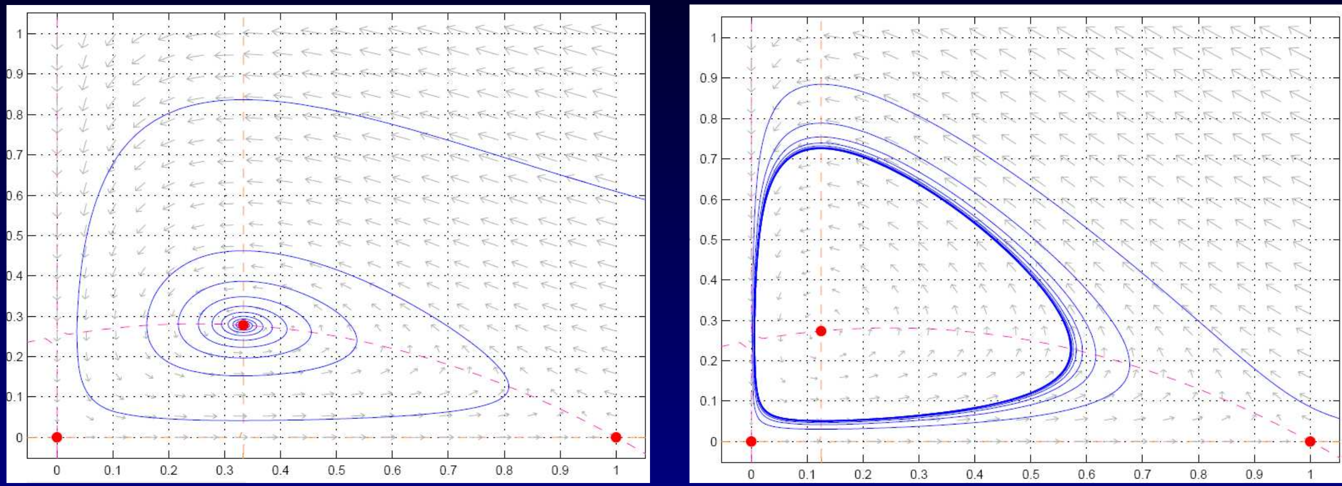
Classical examples: Hudson company lynx-hare data in 1800s, Volterra model

Alfred Lotka (1880-1949) Vito Volterra (1860-1940)

$$\frac{du}{dt} = u(a - bu) - cuv, \quad \frac{dv}{dt} = -dv + fuv.$$



Bifurcation in predator-prey model



<u>Paradox of enrichment</u>:

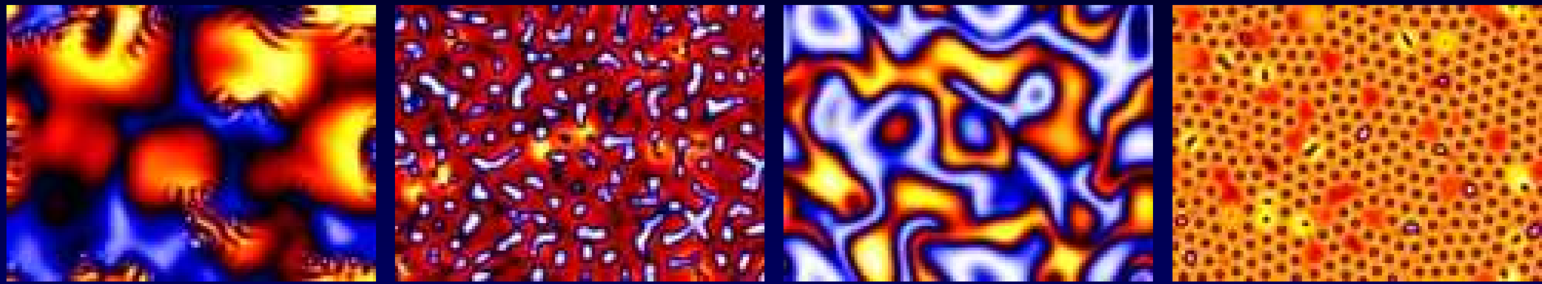
$$\frac{dU}{ds} = \gamma U \left(1 - \frac{U}{K}\right) - \frac{CMUV}{A+U}, \quad \frac{dV}{ds} = -DV + \frac{MUV}{A+U}.$$

Environment is enriched if K (carrying capacity) is larger, but when K is small, a coexistence equilibrium is stable; but when K is larger, the coexistence equilibrium is unstable, and a stable periodic solution appears.

[Rosenzweig, 1971] Paradox of enrichment: destabilization of exploitation ecosystems in ecological time. *Science* **171**.

[May, 1972] Limit cycles in predator-prey communities. *Science*, **177**.

Rich spatial patterns in diffusive predator-prey system



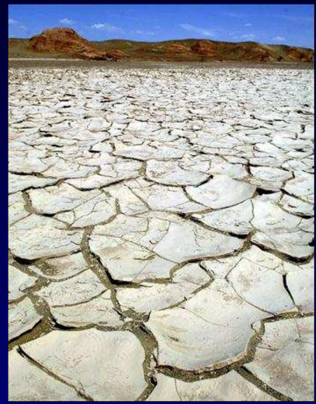
Patterns generated by diffusive predator-prey system

$$\begin{cases} u_t - d_1 \Delta u = u(1-u) - \dfrac{muv}{u+a}, & x \in \Omega,\ t>0, \\ v_t - d_2 \Delta v = -\theta v + \dfrac{muv}{u+a}, & x \in \Omega,\ t>0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega,\ t>0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Patchiness (spatial heterogeneity) of plankton distributions in phytoplankton-zooplankton interaction

[Medvinsky, Li, et.al., 2002] Spatiotemporal complexity of plankton and fish dynamics. *SIAM Review* **44**.

Spatial model: Bifurcation of grassland to desert

$$\frac{\partial w}{\partial t} = a - w - wn^2 + \gamma \frac{\partial w}{\partial x}, \quad \frac{\partial n}{\partial t} = wn^2 - mn + \Delta n, \quad x \in \Omega.$$

$w(x, y, t)$:concentration of water; $n(x, y, t)$: concentration of plant,

Ω: a two-dimensional domain.

$a > 0$: rainfall; $-w$: evaporation; $-wn^2$: water uptake by plants; water flows downhill at speed γ; wn^2: plant growth; $-mn$: plant loss

[Klausmeier, 1999] Regular and Irregular Patterns in Semiarid Vegetation. *Science* **284**.

[Rietkerk, et.al. 2004] Self-Organized Patchiness and Catastrophic Shifts in Ecosystems. *Science* **305**.

Models of differential (difference) equations

Ordinary differential equations: $\dfrac{dy}{dt} = f(\lambda, y), \quad y \in \mathbf{R}^n$

Matrix model: $A(k+1) = f(\lambda, A(k)), \quad A(k) \in \mathbf{R}^n$

Partial differential equations (reaction-diffusion systems):

$$\begin{cases} u_t = d_1 \Delta u + f(\lambda, u, v), & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v + g(\lambda, u, v), & x \in \Omega,\ t > 0, \end{cases}$$

Other effect: delay, non-local, age-structure, etc.

Delay Reaction-diffusion equation:
$u_t(x,t) = d\Delta u(x,t) + f(\lambda, u(x,t), u(x, t-\tau))$

Non-local Reaction-diffusion equation:
$u_t(x,t) = d\Delta u(x,t) + u(x,t)(1 - \int_\Omega f(x,y)u(y,t)dy)$

Steady state solutions

Steady state solutions: solution independent of time t

Ordinary differential equations: $f(\lambda, y) = 0, \quad y \in \mathbf{R}^n$

Matrix model: $A = f(\lambda, A), \quad A \in \mathbf{R}^n$

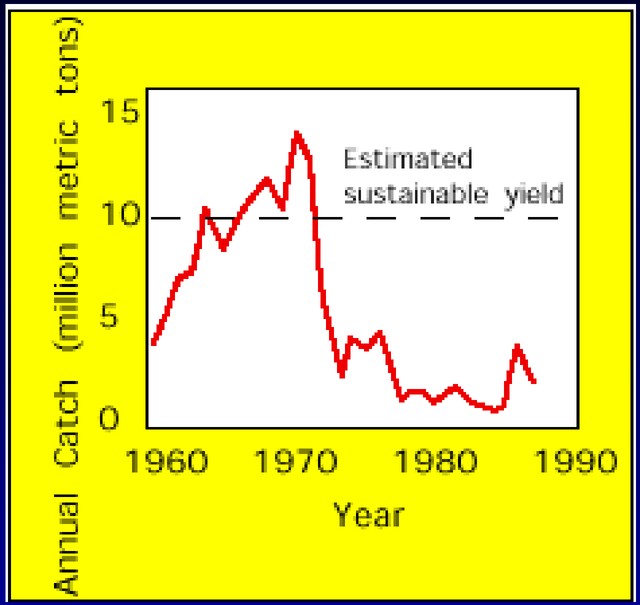
Partial differential equations (reaction-diffusion systems):

$$\begin{cases} d_1 \Delta u + f(\lambda, u, v) = 0, & x \in \Omega, \\ d_2 \Delta v + g(\lambda, u, v) = 0, & x \in \Omega, \end{cases}$$

Abstract form: $F(\lambda, u) = 0$, $u \in X$ (state space, or phase space for the dynamics)

ODE and matrix models: $X = \mathbf{R}^n$,
PDE: $X =$ (infinite dimensional) function space ($W^{2,p}(\Omega)$, $C^{2,\alpha}(\overline{\Omega})$,etc.)
In general X is a Banach space

Bifurcation in R^1 (saddle-node bifurcation)



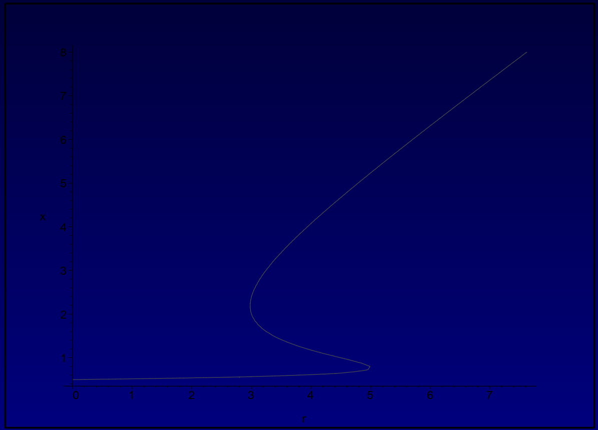
Annual catch of the Peruvian Anchovy Fishery from 1960-1990

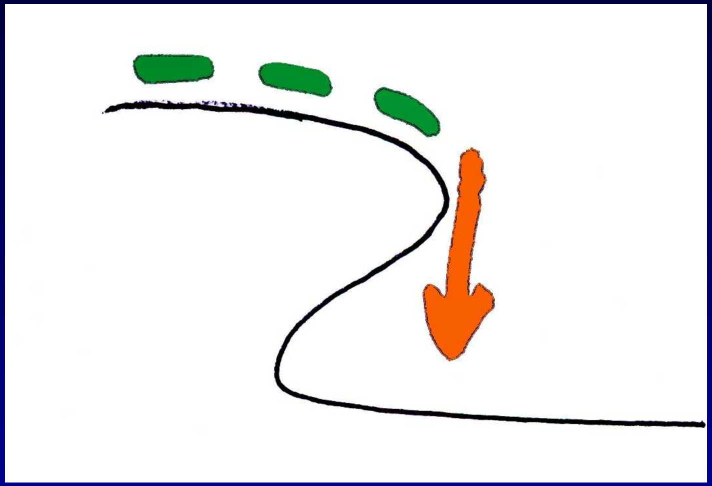
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - H, \quad \text{steady state: } kP\left(1 - \frac{P}{N}\right) - H = 0$$

When $H > H_0 \equiv \dfrac{kN}{4}$, the fishery collapses.

H_0 is the maximum sustainable yield (MSY)

Bifurcation in $\mathbb{R}^1$ (hysteresis)





A grazing system of herbivore-plant interaction

$$\frac{dV}{dt} = V(1-V) - \frac{rV^p}{h^p + V^p}, \quad h, r > 0, \ \ p \geq 1.$$

[Noy-Meir, 1975] Stability of Grazing Systems: An Application of Predator-Prey Graphs. *J. Ecology* **63**.

[May 1975] Thresholds and breakpoints in ecosystems with a multiplicity of stable states. *Nature* **269**.

Catastrophe theory: Thom, Arnold, Zeeman in 1960-70s

Potential catastrophes: Arctic sea, Greenland ice, Amazon rainforest, etc.

Nonlinear equations in Banach space and derivatives

Equation: $F(\lambda, u) = 0$, $F : \mathbf{R} \times X \to Y$
Example: $\Delta u + \lambda f(u) = 0$, $u \in X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and $Y = L^p(\Omega)$.

Partial derivatives: (Frechét derivative)

$$F_u(\lambda, u)[\phi] = \lim_{h \to 0} \frac{F(\lambda, u + h\phi) - F(\lambda, u)}{h}$$

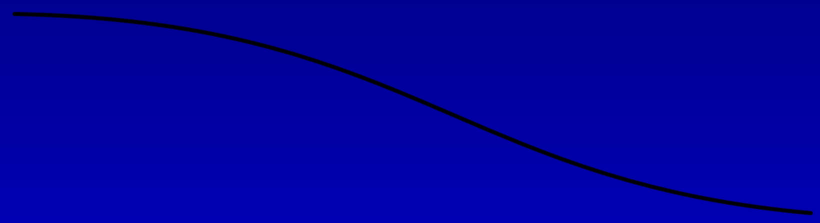
$$F_\lambda(\lambda, u) = \lim_{h \to 0} \frac{F(\lambda + h, u) - F(\lambda, u)}{h}$$

If $F(\lambda, u) = \Delta u + \lambda f(u)$, then
$F_u(\lambda, u)[\phi] = \Delta\phi + \lambda f'(u)\phi$,
$F_\lambda(\lambda, u) = f(u)$.

$F_u(\lambda, u)$ is a linear mapping from X to Y
$N(F_u(\lambda, u)) \subset X$ is the null space (the space of solutions of $F_u(\lambda, u)[\phi] = 0$)
$R(F_u(\lambda, u)) \subset Y$ is the range space

Implicit function theorem: no bifurcation

Theorem 0 Assume X, Y are Banach spaces. Let $(\lambda_0, u_0) \in \mathbf{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y. Let $F(\lambda_0, u_0) = 0$ and $F_u(\lambda_0, u_0)$ is invertible ($F_u(\lambda_0, u_0)[\phi] = 0$ only has zero solution). Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $(\lambda, u(\lambda))$, $u(\lambda) = u_0 + (\lambda - \lambda_0)w_0 + z(\lambda)$, where $w_0 = -[F_u(\lambda_0, u_0)]^{-1}(F_\lambda(\lambda_0, u_0))$ and $\lambda \mapsto z(\lambda) \in X$ is a continuously differentiable function near $s = 0$ with $z(0) = z'(0) = 0$.



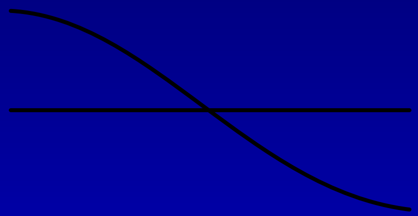
Basic Bifurcation in R^1

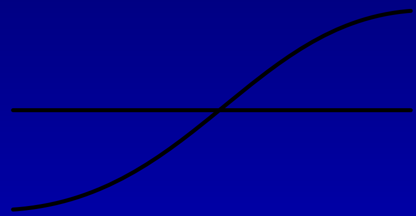
Consider

$$f(\lambda, u) = 0, \quad \lambda \in \mathbf{R}, u \in \mathbf{R}.$$

Assume $f(\lambda, u_0) = 0$ for $\lambda \in \mathbf{R}$, and $f_u(\lambda_0, u_0) = 0$.
Transcritical bifurcation:

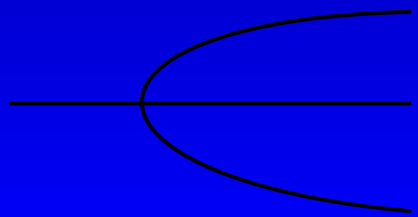
$$f_{\lambda u}(\lambda_0, u_0) \neq 0, \text{ and } f_{uu}(\lambda_0, u_0) \neq 0.$$

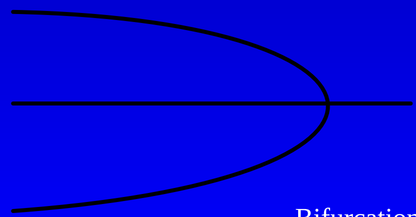




Pitchfork Bifurcation

$$f_{uu}(\lambda_0, u_0) = 0, \ f_{\lambda u}(\lambda_0, u_0) \neq 0, \text{ and } f_{uuu}(\lambda_0, u_0) \neq 0.$$





Saddle-node bifurcation theorem

Theorem 1 [Crandall-Rabinowitz, 1973]

Let $F : \mathbf{R} \times X \to Y$ be continuously differentiable. $F(\lambda_0, u_0) = 0$, F satisfies

(F1) $dimN(F_u(\lambda_0, u_0)) = codimR(F_u(\lambda_0, u_0)) = 1$, and

(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a continuously differentiable curve $(\lambda(s), u(s))$, $(\lambda(0), u(0) = (\lambda_0, u_0)$, $\lambda'(0) = 0$ and $u'(0) = w_0$.

based on Implicit function theorem

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based on Implicit function theorem

We will later consider the case

(F2') $F_\lambda(\lambda_0, u_0) \in R(F_u(\lambda_0, u_0))$.

Transcritical-Pitchfork bifurcation theorem

Theorem 2[Crandall-Rabinowitz, 1971]

Let $F : \mathbf{R} \times X \to Y$ be continuously differentiable. Suppose that $F(\lambda, u_0) = 0$ for $\lambda \in \mathbf{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At (λ_0, u_0), F satisfies

(F1) $dim N(F_u(\lambda_0, u_0)) = codim R(F_u(\lambda_0, u_0)) = 1$, and

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$, where $w_0 \in N(F_u(\lambda_0, u_0))$,

Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $(\lambda(s), u(s))$, $s \in I = (-\delta, \delta)$, where $(\lambda(s), u(s))$ are continuously differentiable functions such that $\lambda(0) = \lambda_0$, $u(0) = u_0$, $u'(0) = w_0$.

$N(F_u(\lambda_0, u_0))$: the null space, $R(F_u(\lambda_0, u_0))$: the range space

[Chow-Hale, 1982] [Deimling, 1985]

Assumptions

Assume that $F(\lambda_0, u_0) = 0$, F satisfies

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Decomposition of spaces:

$X = N(F_u(\lambda_0, u_0)) \oplus Z$

$Y = R(F_u(\lambda_0, u_0)) \oplus Y_1$

$w_0 (\neq 0) \in N(F_u(\lambda_0, u_0))$

$F_u(\lambda_0, u_0)|_Z : Z \to R(F_u(\lambda_0, u_0))$ is an isomorphism

there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v \rangle = 0\}$

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there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v \rangle = 0\}$

There exists a unique $v_1 \in Z$ such that $F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[v_1] = 0$.

Crossing curve bifurcation theorem

Theorem 3 [Liu-Wang-Shi, 2007]
Let $F : \mathbf{R} \times X \to Y$ be a C^2 mapping, and assume conditions above. In addition the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H_0 \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1, v_1]\rangle & \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \\ \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle & \langle l, F_{uu}[w_0, w_0]\rangle \end{pmatrix}$$

is non-degenerate, i.e., $\det(H_0) \neq 0$. Let S be the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$.

1. If H_0 is definite, *i.e.* $\det(H_0) > 0$, then S is the singleton $\{(\lambda_0, u_0)\}$.
2. If H_0 is indefinite, *i.e.* $\det(H_0) < 0$, then S is the union of two intersecting C^1 curves, which are in form of $(\lambda_i(s), u_i(s)) = (\lambda_0 + \mu_i s + s\theta_i(s), u_0 + \eta_i s w_0 + s v_i(s))$, $i = 1, 2$, where $s \in (-\delta, \delta)$ for some $\delta > 0$, (μ_i, η_i) are non-zero linear independent solutions of the equation

Crossing curve bifurcation theorem (cont.)

$$\langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1, v_1]\rangle \mu^2 + 2\langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \eta\mu$$
$$+ \langle l, F_{uu}[w_0, w_0]\rangle \eta^2 = 0,$$

$\theta_i(0) = \theta_i'(0) = 0$, $v_i(s) \in Z$, and $v_i(0) = v_i'(0) = 0$, $i = 1, 2$.

Crossing curve bifurcation theorem (cont.)

$$\langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1, v_1]\rangle \mu^2 + 2\langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \eta\mu$$
$$+ \langle l, F_{uu}[w_0, w_0]\rangle \eta^2 = 0,$$

$\theta_i(0) = \theta_i'(0) = 0$, $v_i(s) \in Z$, and $v_i(0) = v_i'(0) = 0$, $i = 1, 2$.

Remarks:

1. When $F_\lambda(\lambda_0, u_0) = 0$, we have $v_1 = 0$.

$$H_1 \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle & \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle \\ \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle & \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle \end{pmatrix}$$

and the equation of tangents of curves become

$$\langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle \mu^2 + 2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle \mu\eta + \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle \eta^2 = 0.$$

More remarks

2. If $F(\lambda, u_0) \equiv 0$, then Theorem 2 (classical transcritical and pitchfork bifurcation theorem) follows from Theorem 3.

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3. Theorem 3 is a natural complement Crandall-Rabinowitz saddle-node bifurcation theorem (Theorem 2), where $(F2)$ is imposed. Our result is based on condition the opposite $(F2')$ and a generic second order non-degeneracy condition $\det(H_0) \neq 0$.

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4. We also prove a secondary bifurcation theorem, which generalizes the one in [Crandall-Rabinowitz, 1971] and [Deimling, 1985]. In the previous ones, a solution curve Γ_1 is given, and it is shown that another curve Γ_2 exists and intersects with Γ_1 transversally. In our result, no any solution curve is given, and we obtain the two curves simultaneously.

Summary of Bifurcation Theorems

Let $F : \mathbf{R} \times X \to Y$ be continuously differentiable. $F(\lambda_0, u_0) = 0$, F satisfies

(F1) $dimN(F_u(\lambda_0, u_0)) = codimR(F_u(\lambda_0, u_0)) = 1$, and

(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

Then a saddle-node bifurcation occurs.

If F satisfies **(F1)**,

(F2') $F_\lambda(\lambda_0, u_0) \in R(F_u(\lambda_0, u_0))$,

and additional non-degeneracy condition on D^2F

Then a crossing curve bifurcation occurs. (include pitchfork and transcritical bifurcations)

What if kernel is 2-dimensional?

Let $F : \mathbf{R} \times X \to Y$ be continuously differentiable. $F(\lambda_0, u_0) = 0$, F satisfies

(F1-2) $dimN(F_u(\lambda_0, u_0)) = codimR(F_u(\lambda_0, u_0)) = 2$, and

(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

and additional non-degeneracy condition on $D^2 F$

Then a saddle-node bifurcation of two curves occurs
[Liu, Shi, Wang, preprint]

If F satisfies **(F1-2)**,

(F2') $F_\lambda(\lambda_0, u_0) \in R(F_u(\lambda_0, u_0))$,

more complicated, depending on the symmetry of the problem
(example: two-dimensional surface, four curves)

Proof of main result

1. Lyapunov-Schmidt reduction: reduce $F(\lambda, u) = 0$ to
$G(\lambda, t) \equiv \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$
2. A finite dimensional result (improving result based on Morse lemma)

Morse lemma: [Nirenberg, 1974] Suppose that $f : \mathbf{R}^k \to \mathbf{R}$ is a C^p function, $k \geq 2$. If $f(0) = 0$, $f_x(0) = 0$, and the Hessian $f_{xx}(0)$ is a non-degenerate $k \times k$ matrix. Then there exists a local C^{p-2} coordinate change $y(x)$ defined in a neighborhood of the origin with $y(0) = 0$, $y_x(0) = I$ such that

$$f(x) = \frac{1}{2} y(x)^T f_{xx}(0) y(x),$$

where $y(x)^T$ is the transpose of $y(x)$, and $y(x)$ is assumed to be column vector in $\mathbf{R}^k$. In particular if $k = 2$ and $f_{xx}(0)$ is indefinite, then the set of solutions of $f(x) = 0$ near the origin consists of two C^{p-2} curves intersecting only at the origin.

A finite dimensional theorem

[Liu-Wang-Shi, 2007]

Suppose that $(x_0, y_0) \in \mathbf{R}^2$ and U is a neighborhood of (x_0, y_0). Assume that $f : U \to \mathbf{R}$ is a C^p function for $p \geq 2$, $f(x_0, y_0) = 0$, $\nabla f(x_0, y_0) = 0$, and the Hessian $H = H(x_0, y_0)$ is non-degenerate. Then

1. If H is definite, then (x_0, y_0) is the unique zero point of $f(x, y) = 0$ near (x_0, y_0);
2. If H is indefinite, then there exist two C^{p-1} curves $(x_i(t), y_i(t))$, $i = 1, 2$, $t \in (-\delta, \delta)$, such that the solution set of $f(x, y) = 0$ consists of exactly the two curves near (x_0, y_0), $(x_i(0), y_i(0)) = (x_0, y_0)$. Moreover t can be rescaled and indices can be rearranged so that $(x_1'(0), y_1'(0))$ and $(x_2'(0), y_2'(0))$ are the two linear independent solutions of

$$f_{xx}(x_0, y_0)\eta^2 + 2f_{xy}(x_0, y_0)\eta\tau + f_{yy}(x_0, y_0)\tau^2 = 0.$$

Proof of the "calculus problem"

Consider

$$x' = \frac{\partial f(x,y)}{\partial y}, \quad y' = -\frac{\partial f(x,y)}{\partial x}, \quad (x(0), y(0)) \in U.$$

Then it is a Hamiltonian system with potential function $f(x,y)$, (x_0, y_0) is the only equilibrium point in U, and (x_0, y_0) is a saddle point. From the invariant manifold theory of differential equations, the set $\{f(x,y) = 0\}$ near (x_0, y_0) is consisted of the 1-dimensional stable and unstable manifolds at (x_0, y_0), which are C^{p-1} since f is C^p.

Question: is there a proof without using invariant manifold theory but only elementary calculus?

another more general theorem:
splitting lemma [Kuiper, 1972] [Chang, 1993] [Li, Li, Liu, 2005]

Global bifurcation

Theorem 4 [Rabinowitz, 1971]

Suppose that L is a compact operator on X, and $H(\lambda, u)$ is a compact operator on $\mathbf{R} \times X$. If λ_0 is a characteristic value of L with odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point of $F(\lambda, u) \equiv u - \lambda L u - H(\lambda, u) = 0$. Moreover, if Σ is the set of the nontrivial solutions of $F(\lambda, u) = 0$, then there is a closed connected component Σ_1 of $\overline{\Sigma}$, such that $(\lambda_0, 0) \in \Sigma_1$, and either (i) Σ_1 is unbounded; or (ii) Σ_1 contains $(\lambda_*, 0)$, where $\lambda_* (\neq \lambda_0)$ is also a characteristic value of L.

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It requires strong compactness. For applications in PDEs, it usually requires to take inverse of Δ operators or more general elliptic operators. For some applications with cross-diffusion or nonlinear boundary conditions, taking inverse operators are not easy.

Global bifurcation from simple eigenvalue theorem

Theorem 5 [Crandall-Rabinowitz, 1971]

Let $F : \mathbf{R} \times X \to Y$ be continuously differentiable. Suppose that $F(\lambda, u_0) = 0$ for $\lambda \in \mathbf{R}$, the partial derivative $F_{\lambda u}$ exists and is continuous. At (λ_0, u_0), F satisfies

(F1) $dim N(F_u(\lambda_0, u_0)) = codim R(F_u(\lambda_0, u_0)) = 1$, and

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$, where $w_0 \in N(F_u(\lambda_0, u_0))$,

Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $(\lambda(s), u(s))$, $s \in I = (-\delta, \delta)$, where $(\lambda(s), u(s))$ are C^1 functions such that $\lambda(0) = \lambda_0$, $u(0) = u_0$, $u'(0) = w_0$.

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[Pejsachowicz-Rabier, 1998] [Shi-Wang, 2008]
If in addition, $F_u(\lambda, u)$ is a Fredholm operator for all $(\lambda, u) \in \mathbf{R} \times X$, then the curve $\{(\lambda(s), u(s)) : s \in I\}$ is contained in $\mathcal{C}$, which is a connect component of $S = \{(\lambda, u) \in \mathbf{R} \times X : F(\lambda, u) = 0, u \neq u_0\}$; and either $\mathcal{C}$ is not compact, or $\mathcal{C}$ contains a point $(\lambda_*, 0)$ with $\lambda_* \neq \lambda_0$.

Fredholm operators of index zero

Quasilinear elliptic systems with nonlinear boundary conditions are Fredholm operators of index zero

Theorem 6 [Shi-Wang, 2008]
Suppose that $p > n$, $\partial\Omega \in C^3$, and the regularity assumption above holds. Let U be an open connected set of $\mathbf{R} \times (W^{2,p}(\Omega))^N$. Assume that for each fixed $(\lambda, u) \in U$, $D_uT(\lambda, u) = (D_uA(\lambda, u), D_uB(\lambda, u))$ is elliptic on $\overline{\Omega}$, and that for a particular $(\lambda_0, u_0) \in U$, $D_uT(\lambda_0, u_0)$ satisfies Agmon's condition at a θ_0, then the Fredholm index of $D_uT(\lambda, u)$ is 0 for all $(\lambda, u) \in U$.

It will have many applications in reaction-diffusion systems in mathematical biology, physics, and chemistry.

Example 1

Cross-diffusion system:

$$\begin{cases} \Delta[(1+\alpha_1 u+\alpha_2 v)u]+u(\lambda-u-bv)=0, & x\in\Omega, \\ \Delta[(1+\beta_1 u+\beta_2 v)v]+v(\mu+cu-v)=0, & x\in\Omega, \\ u=v=0, & x\in\partial\Omega. \end{cases}$$

Competing species with passive diffusion, self-diffusion, cross-diffusion.
[Shigesada, Kawasaki and Teramoto, 1979]
[Nakashima, Yamada, 1996] [Kuto, Yamada, 2004]: $\alpha_1=\beta_2=0$
Their idea: $U=(1+\alpha_2 v)u$, $V=(1+\beta u)v$, then the system becomes semilinear but with messy nonlinearities.

We prove the existence of a bounded branch of coexistence solutions which connecting the two semi-trivial solution branches via our new global bifurcation theorem. Our method is definitely more direct.

Example 2

chemotactic diffusion system:

$$\begin{cases} u'' - f(u)v = 0 & x \in (0,1), \\ \lambda v'' - \chi(v\psi'(u)u')' + (kf(u) - \theta - \beta v)v = 0 & x \in (0,1), \\ u'(0) = 0, \;\; u'(1) = h(1 - u(1)), & \\ \lambda v' - \chi v\psi'(u)u' = 0 \quad \text{at } x = 0, 1. & \end{cases}$$

[Wang, 2000] Idea: make an inverse of the main part of the differential operator, then use Rabinowitz's global bifurcation theorem.

We directly apply the new global bifurcation theorem for quasilinear systems.

Final Remark

If we do not succeed in solving a mathematical problem then, very often, the reason is that we did not yet discover the more general point of view from which the given problem appears to be a link in a chain of related problems. Having found this point of view, not only the given problem becomes more accessible to our research, but we also gain a method which is applicable to related problems $\cdots$

In dealing with mathematical problems, specializing plays—as I believe—an even more important role than generalizing. Perhaps in most cases in which we fail to find an answer, the reason for this failure is that we did not solve, at least not completely, problems simpler and easier than the given one. Everything amounts to finding these easier problems and to solve them by using tools which are as perfect as possible and concepts which are fit to be generalized.

David Hilbert, lecture at the International Congress of Mathematicians in Paris, 1900

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Thank You