Review of Multi-variable calculus:

The functions in all models depend on two variables: time $t$ and spatial variable $x$, $(x, y)$ or $(x, y, z)$.

The spatial variable represents the environment where the species is living (bacteria: tank in lab, rabbits and foxes: woods, birds: the space).

The time variable is one dimension, we call it time interval. Usually it is $(-\infty, \infty)$, $[0, \infty)$ or $[0, T]$.

In mathematics we call the environment spatial domain (or simply domain), or region.
Domains

The choice of domain in a model depends on the nature of the problem.

Most of time, domain is bounded. (lab tank, woods, island, earth, universe?). And it has a boundary.

Mathematically we assume that a bounded domain is an interval $(a, b)$ in 1-d, the region enclosed by a circular curve in 2-d, or the region enclosed by a spherical surface in 3-d.

Sometime for simplicity, or to observe certain phenomenon clearer, we also consider the whole space $\mathbb{R} = (−\infty, \infty)$, $\mathbb{R}^2$ or $\mathbb{R}^3$.

We will call a domain $\Omega$. 
Functions

Functions in the models are defined for (time interval $\times$ domain).

Let $X$ be $x$, $(x, y)$ or $(x, y, z)$. Then the function is in a form of $f(t, X)$.

Example: Let $D$ be a 2-d domain. (a woods)
$R(t, x, y)$ = the density of rabbit population at location $(x, y)$ and time $t$
$F(t, x, y)$ = the density of fox population at location $(x, y)$ and time $t$

Population density = \[
\frac{\text{total population in an area}}{\text{area}}
\]

Example: population density is 50,000 per square kilometer in NYC, and it is 5,000 in Williamsburg
**Graph of the function:** (hard to draw in 2-d or 3-d)

graph: \((x, y, f(x, y))\) (Maple), \((x, y, z, f(x, y, z))\).

level curve (contour): the graph of \(f(x, y) = c\). (Maple)

level surface: the graph of \(f(x, y, z) = c\). (Maple)

Derivatives: partial derivatives
\[
\frac{\partial f(t, x, y)}{\partial t} = f_t, \quad \frac{\partial f(t, x, y)}{\partial x} = f_x
\]

Gradient: \(\nabla f(x, y) = (\partial f/\partial x, \partial f/\partial y)\)

Gradient at one point is a vector; gradient function is a vector field; gradient vector is perpendicular to the level curve
Vector field: (a vector) \( F(x, y) = (f(x, y), g(x, y)) \)

Jacobian: (a matrix) \( J = \begin{pmatrix}
f_x(x, y) & f_y(x, y) 
g_x(x, y) & g_y(x, y)
\end{pmatrix} \)

Divergent of a vector field: (a scalar)
for \( F(x, y) = (f(x, y), g(x, y)) \), \( \text{div}(F) = f_x + g_y \)

Laplacian of a function: (a scalar)
for a function \( f(x, y) \), \( \Delta f = \text{div}(\nabla f) = \text{div}(f_x, f_y) = f_{xx} + f_{yy} \)

Hessian of a function: (a matrix)
for a function \( f(x,y) \), Jacobian of \( \nabla f \), \( H = \begin{pmatrix}
f_{xx}(x, y) & f_{xy}(x, y) 
f_{yx}(x, y) & f_{yy}(x, y)
\end{pmatrix} \)

Example: \( f(x, y) = x^2 + 2y^2 - 2xy \).
(1) Find \( \nabla f \); (2) Find Hessian of \( f \); (3) Find \( \Delta f \).
Different kinds of functions:

\( P(t) \): function (one variable, one function)

\( P(x, y) \): multi-variable function (two variables, one function)

\( (P(t), Q(t)) \): vector valued function (one variable, two functions)

\( (P(x, y), Q(x, y)) \): vector field (two variables, two functions)
Integral of functions: \( \Omega \): two-dimensional domain, boundary \( \partial \Omega \) a closed curve, \( X = (x, y) \)

\[
\int_{\Omega} f(x, y) \, dX = \int \int_{\Omega} f(x, y) \, dx \, dy \quad \int_{\Omega} 1 \, dX = \text{area of } \Omega
\]

Divergence Theorem:

Let \( \vec{F}(x, y) \) be a vector field, and let \( \vec{n}(x, y) \) be the unit outer normal vector at \( (x, y) \), a boundary point on \( \partial \Omega \). Then \( \int_{\partial \Omega} \vec{F}(x, y) \cdot \vec{n}(x, y) \, ds \) is the total flux of \( \vec{F} \) over the curve \( \partial \Omega \).

\[
\int_{\partial \Omega} \vec{F}(x, y) \cdot \vec{n}(x, y) \, ds = \int_{\Omega} \text{div}(\vec{F}(x, y)) \, dX.
\]

1-d: \( F(b) - F(a) = \int_{a}^{b} F'(x) \, dx \)
Green’s Identities:

\[ \int_{\Omega} u \Delta v dX = \int_{\partial \Omega} u \nabla v \cdot \vec{n} ds - \int_{\Omega} \nabla u \cdot \nabla v dX \]

\[ \int_{\Omega} u \Delta v dX - \int_{\Omega} v \Delta u dX = \int_{\partial \Omega} u \nabla v \cdot \vec{n} ds - \int_{\partial \Omega} v \nabla u \cdot \vec{n} ds \]

Example: Let \( F(x, y) = (x + y, e^{x-y}) \), and let \( \Omega \) be a square \((0, 1) \times (0, 1)\).

(1) Calculate \( \int_{\Omega} \text{div}(\vec{F}(x, y)) dX \)

(2) calculate \( \int_{\partial \Omega} \vec{F}(x, y) \cdot \vec{n}(x, y) ds \)